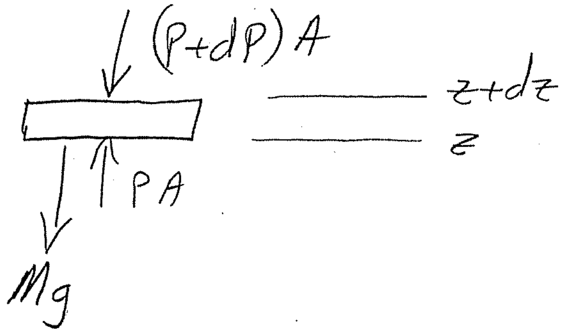
Hydrostatic Equilibrium

Balance the vertical forces of pressure and gravity on a packet of gas in the atmosphere.



A = Area
P = Pressure
M = mass
g = gravity
V = Volume

Forces up = Forces Down

$$PA = Mg + (P+dP)A$$

$$A\,dP = -Mg \qquad \text{use} \quad M = \rho V = \rho(A\,dz)$$

$$\boxed{dP = -\rho g\,dz}$$

In general, $\rho = \rho(z)$
$g = g(z)$

here we assume g = constant and use the ideal gas law to write ρ in terms of P

$\boxed{K = \text{Boltzmann's Constant}}$

$$\text{so} \quad PV = NKT \Rightarrow \rho = \frac{M}{V} = \frac{MP}{NKT}$$

$$dP = -\frac{MP}{NKT}\,g\,dz \qquad \text{Now define } m = \frac{M}{N}$$

m = mean molecular mass (with units of kg)

$$\Rightarrow \frac{dP}{P} = -\frac{mg}{KT}\,dz$$

Integrate.

$$\int \frac{dP}{P} = \int -\frac{mg}{kT}\, dz$$

since g is constant, it comes out of the integral
we also assume T = constant and u = const.

$$\ln P = -\frac{mg}{kT} z + K$$

where K is the constant of integration

or

$$P = e^{K} \cdot e^{-\frac{mg}{kT} z}$$

let $P_0 = e^{K}$
and $H = \frac{kT}{mg}$

$$\boxed{P = P_0\, e^{-z/H}}$$

H is the scale height of the atmosphere
One scale height above a reference level, P_0,
the atmospheric pressure is only $\frac{P_0}{e} = 0.37 P_0$
The density ρ also falls off exponentially
since $\rho = \frac{m}{kT} P$ with m, k, T assumed
constant.