1. Quantum Principles

So far, we have considered astrophysical systems and processes that can be understood using classical physics – i.e., physics as it was developed at the end of the 19th century. Of course, while the physical principles – i.e., the basic sets of equations – had been enunciated, the application of these principles to understand complex (and even simple!) astronomical systems had only developed to a limited extent at that point. Much of contemporary theoretical astrophysics is still primarily based on classical physics, although in most areas there is a mixture of "classical" and "quantum" that is required.

The quantum "revolution" that took place in physics in the first ~ 25 years of the 20th century also profoundly changed astronomy. With quantum theory, it was understood why astronomical objects emit spectra that have "lines", and it became possible to diagnose the conditions (e.g., ρ, T) within astronomical bodies based on the relative strengths of observed lines. In the 1920s and 1930s, the understanding of how nuclear fusion – a quantum process – powers stars was developed. In the 1930s, it was also realized that quantum effects are crucial for understanding the possible pathways of stellar evolution. Ultimately – but not yet! – the nature of dark matter and dark energy may be revealed based on quantum physics.

Given the limited time we have, we cannot here provide a full development of quantum theory. Instead, we will introduce some of the basic principles, in order to be able to use them for astrophysical applications. The main applications we will consider are to stellar structure; this includes understanding the regulation of fusion within the cores of stars.

One of the key developments of quantum theory was Einstein's 1905 proposal that all light consists of energy "quanta", based (in part) on his realization that the threshold frequency behavior of the photoelectric effect could be explained if light is quantized. Einstein reasoned that a minimum energy was needed to liberate electrons and produce a current when light is shone on a metal, and that the observed minimum frequency of light that was required could be explained if there is a relationship between packets of energy and frequency. This relationship is $E = h\nu$. Since light is also a wave, it must have *both* wave and particle nature – this is known as "duality". Since $\nu = c/\lambda$, energy is also related to wavelength by $E = hc/\lambda$.

In 1905, in addition to developing the idea of radiation quantization, Einstein also developed the theory of special relativity. One of the key results of special relativity if that energy, momentum, and mass are all related by

$$E^2 - c^2p^2 = m^2c^4 \quad .$$

Since photons are massless, $m = 0$, this implies that $E = cp$, which when combined with $E = h\nu = hc/\lambda$ implies $p = h/\lambda$. Thus a photon's momentum is related to the wavelength of the corresponding wave. (Instead of using λ and $h = 6.63 \times 10^{-27}$ ergs s (Planck's constant) it is convenient to use $\hbar = h/2\pi = 1.05 \times 10^{-27}$ ergs s and the wavenumber $k = 2\pi/\lambda : \; p = \hbar k$.)

Another key development in quantum theory was Bohr's proposal (1913) that the energy states of atoms are discrete rather than continuous, and that transitions between states occur by quantized jumps, with the energy difference carried by a photon that is emitted or absorbed.

Bohr's theory was based on the idea that the orbital "action" of an electron in an atom is quantized; this is equivalent to quantization of the orbital angular momentum in an orbital model, $rp = n\hbar$ for some integer n. With this, Bohr showed that

$$E_n = -\frac{m_e e^4}{2\hbar\, n^2}$$

for the hydrogen atom. Bohr's model successfully explained explained what was then known – but not understood – about spectral lines following certain patterns.

In 1924, de Broglie introduced a key idea that underlies all of quantum theory. This idea is that all matter, not just light, has a dual wave-particle nature, with free particles having effective wavelengths $\lambda = h/p$, and hence $\hbar k = \hbar\, 2\pi/\lambda = \hbar\, 2\pi(p/h) = p$, or, using vectors

$$\vec{p} = \hbar\, \vec{k} \quad .$$

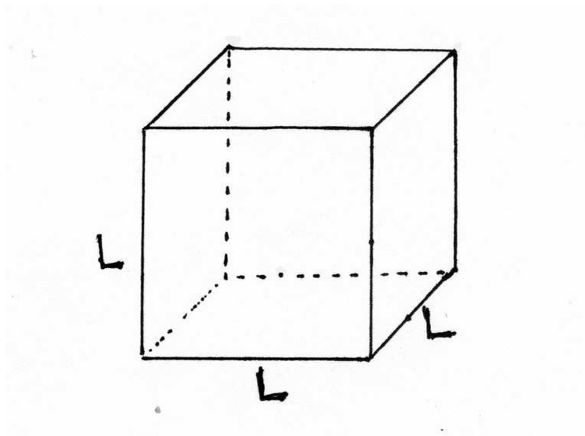
This proposal was experimentally verified by the electron diffraction experiments of Davisson and Germer (1927).

Following de Broglie, Heisenberg (1925) and Schrödinger (1926) developed quantum mechanics, in which positions and momenta of particles are not unique, but instead obey probability distributions that can be calculated. Schrödinger's technical approach using wavefunctions to describe probability densities became standard. The Schrödinger equation describes how wavefunctions evolve, and the possible measurable values of observables are eigenvalues of the corresponding operators, with any wavefunction a superposition of eigenfunctions of the operators representing observables. (This is all fully explained in Physics 401 and 402.)

After the development of quantum mechanics, Dirac (1928) extended the theory to include special relativity. Subsequent developments led to quantum electrodynamics and quantum field theory, which are covered in graduate-level physics courses.

1.1. Free Particles

Since much of the matter in the universe is in the form of free particles in a plasma, and photons are also free particles, it is essential to understand the implications of quantum theory for distributions of free, independent particles. Even though the particles are free, it is always necessary to consider some particular volume, since we will always describe the end result in terms that include "per unit volume". So, we'll consider a particle within a cube of sides L; the value of L will not appear in the final result.



The particle is taken to have momentum $\vec{p} = (p_x, p_y, p_z)$. According to de Broglie, the corresponding wavenumber with this value of momentum is

$$\vec{k} \;=\; \frac{\vec{p}}{\hbar} \quad .$$

Since, according to Schrödinger, the wavenumber is that corresponding to the probability density of where the particle can be found spatially within the box, the constraint imposed by having walls is that the particle cannot lie inside the walls. It turns out that this is consistent with a probability density that is proportional to

$$P \;\propto\; \sin^2\left(\frac{n_x\pi}{L}x\right)\cdot\sin^2\left(\frac{n_y\pi}{L}y\right)\cdot\sin^2\left(\frac{n_z\pi}{L}z\right) \quad .$$

For this functional form, notice that $P = 0$ at $x = 0$ and L, $y = 0$ and L, $z = 0$ and L. Here, (n_x, n_y, n_z) are a three-tuple of integers

Since this has the same form as $P \propto \sin^2(k_x x)\sin^2(k_y y)\sin^2(k_z z)$, we conclude that the only possible values of $\vec{k}$ are

$$\vec{k} \;=\; \frac{\pi}{L}\,\vec{n} \quad \text{for} \quad \vec{n} \;=\; (n_x, n_y, n_z).$$

and hence

$$\vec{p} \;=\; \hbar\,\vec{k} \;=\; \frac{h}{2L}\,\vec{n} \quad .$$

For massive, nonrelativistic particles, the corresponding energy is

$$E_{NR} \;=\; \frac{|\vec{p}|^2}{2m} \;=\; \frac{\hbar^2|\vec{k}|^2}{2m} \;=\; \frac{\hbar^2}{2m}\left(\frac{\pi}{L}\right)^2\;|\vec{n}|^2 \quad ,$$

while for relativistic particles, the corresponding energy is

$$E_{REL} \;=\; c\;|\vec{p}| \;=\; c\hbar\;|\vec{k}| \;=\; c\hbar\left(\frac{\pi}{L}\right)\;|\vec{n}| \quad .$$

We notice the following important points:

(1) The possible values of the particle energy is not a continuum, but only has certain quantized allowed values. The three-tuple $\vec{n}$ must consist of integers; in-between values are not permitted.

(2) In the limit of a very large box ($L \to \infty$), the possible values of $|\vec{p}|$ and E do approach a continuum, since $\delta|\vec{p}| = (h/2L)\delta|\vec{n}| \to 0$ if $L \to \infty$, even though the *minimum* possible value for $\delta|\vec{n}|$ is unity.

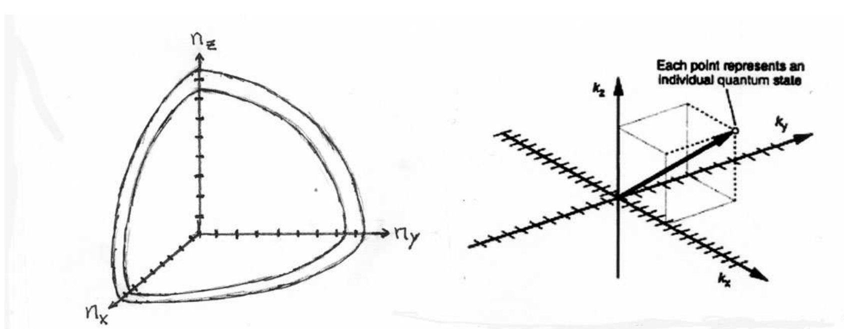
(3) It is possible to have independent states of a particle that nevertheless have the same value of energy, since any three-tuple (n_x, n_y, n_z) that has the same $|\vec{n}|^2 = n_x^2 + n_y^2 + n_z^2$ will have the same energy. These states are different in the sense that the probabilities of where the particle would be found differ. Particle states that are different, but have the same energy, are called "degenerate".

An important question, in terms of assessing particle statistics, is to know, for a given energy, how many different states are degenerate. This is the number of independent quantum particles that can have the same energy.

1.2. Counting States

For particles in a box, we can answer the question of how many particles are degenerate by using a geometrical argument.

Consider a grid in space where each intersection represents a three-tuple (n_x, n_y, n_z). Then a portion of a shell in space that is 1/8 of a sphere, with radius $|\vec{n}|$ and thickness $\delta|\vec{n}| = 1$, will surround all three-tuples that have the same $|\vec{n}|$ and different individual (n_x, n_y, n_z), and therefore represent unique quantum particles.



How many points lie within this 1/8 spherical shell?

$\Delta n = \frac{1}{8} \times$ area of sphere $\times$ thickness of shell $= \frac{1}{8} \times 4\pi|\vec{n}|^2 \times 1 = \frac{1}{8} 4\pi\, n^2\, dn$

(Imagine a cube in rectangular coordinates: say $n_x = 1, 2, 3, 4, 5$, $n_y = 1, 2, 3, 4, 5$ and $n_z = 1, 2, 3, 4, 5$. The number of states is $5 \times 5 \times 5 = 25$ and the volume is also 25, so volume in $\vec{n}$ space = number of states.)

Recall that $\vec{p} = (h/2L)\vec{n}$, so this becomes

$$|\vec{n}| = \left(\frac{2L}{h}\right) |\vec{p}| \quad \Rightarrow \quad \Delta n \;=\; \frac{1}{8}\, 4\pi \left(\frac{2L}{h}\, p\right)^2 d\left(\frac{2L}{h}\, p\right) \;=\; \frac{L^3}{h^3}\, 4\pi\, p^2\, dp$$

Since this is the number of states in the cube of size L, if we want the number in some small volume d^3x, we must multiply by the ratio of volumes, d^3x/L^3, to obtain

$$\boxed{dN \;=\; \frac{d^3x}{h^3}\, 4\pi p^2\, dp \;=\; \frac{1}{h^3}\,(d^3p)(d^3x)}$$

Note that, as promised, the size of our box, L, does not appear.

The total number of states is obtained by integrating over the range of positions and momenta:

$$N \;=\; \int dN \;=\; \frac{1}{h^3} \int\int d^3p\; d^3x$$

Thus, we can interpret this as saying the "size" of a single state in position and momentum is equal to $(\delta x)^3 (\delta p)^3 = h^3$. Any range of momentum and position that is smaller than this could not accommodate a quantum particle. Equivalently, this says that a particle is "localized" within a range $\delta x\, \delta p \sim h$.

Note that the result

$$dN_{spatial\ states} \;=\; \frac{d^3x\; d^3p}{h^3}$$

is just a count for the number of independent <u>spatial</u> states of free quantum particles. If the particles have some other quantity such as <u>spin</u> that is quantized, then we would multiply $dN_{spatial\ states}$ by the number of spin (or other) states per spatial state.

2. Bose-Einstein and Fremi-Dirac Distributions

Although we do not have time here to go into why this comes about, there are two basic types of elementary particles, Bosons and Fermions, which behave differently with respect to other particles due to spin considerations.

Bosons (the most familiar of which are photons) are particles which have integer values of the spin parameter. There is no restriction on how many bosons may be in a given quantum state.

Fermions (including protons, neutrons, and electrons) are particles which have half-integer values of the spin parameter. Fermions are constrained such that only zero or one particle can be in any given quantum state. This is known as the *Pauli Exclusion Principle.*

The difference between Fermion and Boson behavior at the individual particle level has major implications when discussing distributions that contain many particles, since Pauli exclusion effectively means that particles tend to be spread out over more states. In astrophysics, we are particularly interested in the consequences for thermal distributions of particles.

First, let's consider a "gas" of photons in thermal equilibrium. From wave/particle duality, this is equivalent to a thermal distribution of electromagnetic waves.

From the fundamental Boltzmann probability law, we know that the probability of a system having energy ϵ is $P \propto e^{-\epsilon/kT}$. If there are n photons with energy E, then $\epsilon = nE$ and $P \propto e^{-nE/kT}$. So the mean number of photons having a certain energy E is therefore

$$< N(E) > \;=\; \frac{\sum_{n=0}^{\infty} n \cdot P(\epsilon = nE)}{\sum_{n=0}^{\infty} P(\epsilon = nE)} \;=\; \frac{\sum_{n=0}^{\infty} n \cdot e^{-nE/kT}}{\sum_{n=0}^{\infty} e^{-nE/kT}}$$

If we let $x = exp(-E/kT)$, then

$$< N(E) > \;=\; \frac{\sum_{n=0}^{\infty} n \cdot x^n}{\sum_{n=0}^{\infty} x^n}$$

We use the familiar expression for the sum of a power series, $\sum_{n=0}^{\infty} x^n = 1/(1-x)$, and take x times the derivative of both sides to obtain $\sum_{n=0}^{\infty} n \cdot x^n = x/(1-x)^2$. Thus we have $< N(E) >= x/(1-x)$ and thus finally

$$< N(E) > \;=\; \frac{1}{e^{E/kT} - 1} \quad .$$

This is known as the *Bose-Einstein occupation number*, since it is the average occupation expected for a given quantum state. The average occupation number per state can be combined with our previous result for the number of spatial quantum states in a range of positions d^3x and momenta d^3p :

$$dN_{spatial} \;=\; \frac{d^3x \; d^3p}{h^3}$$

Since $E = cp$ for photons, this implies that the number of spatial states per unit volume in a range of energy dE is

$$dn_{spatial} = \frac{dN_{spatial}}{d^3x} = \frac{d^3p}{h^3} = \frac{4\pi\ p^2dp}{h^3} = \frac{4\pi}{c^3}\frac{E^2dE}{h^3}$$

Photons have two possible spin states – independent polarizations – for each spatial state, so

$$dn_{spatial\ \&\ spin} = 2\ \frac{d^3p}{h^3} = \frac{8\pi}{(hc)^3}\ E^2dE$$

Taking all of these factors together, we conclude that the mean number of photons per volume in a thermal distribution with energy between E and dE is

$$d<n> = \frac{8\pi}{(hc)^3}\ E^2dE\ \frac{1}{e^{E/kT}-1} \quad .$$

This can also be written in terms of the frequency $\nu = E/h$; the mean number of photons in a range ν to $\nu + d\nu$ is

$$d<n> = \frac{8\pi}{c^3}\ \frac{\nu^2 d\nu}{e^{h\nu/kT}-1} \quad ;$$

or, in terms of the momentum $p = E/c$, this is

$$d<n> = \frac{8\pi}{h^3}\ \frac{p^2dp}{e^{cp/kT}-1} \quad .$$

This is exactly what was used in an earlier home-work problem to compute the mean number of cosmic background photons per unit volume:

$$<n> = \int d<n> = \frac{8\pi}{(hc)^3}\int_0^\infty \frac{E^2dE}{e^{E/kT}-1}$$

$$= 8\pi\left(\frac{kT}{hc}\right)^3 \int_0^\infty \frac{x^2dx}{e^x-1} = (2.40411)\times 8\pi\left(\frac{kT}{hc}\right)^3$$

The corresponding total radiation density is

$$u_{rad} = \int_0^\infty E\ d<n> = \frac{8\pi}{(hc)^3}\int_0^\infty \frac{E^3dE}{e^{E/kT}-1} = \frac{8\pi}{(hc)^3}\ (kT)^4\int_0^\infty \frac{x^3dx}{e^x-1}$$

$$\text{Since} \quad \int_0^\infty \frac{x^3dx}{e^x-1} = \frac{\pi^4}{15} \quad , \quad u_{rad} = \frac{8\pi^5k^4}{15h^3c^3}\ T^4 = a_{rad}\ T^4$$

where $a_{rad} = 7.5658 \times 10^{-15}$ erg cm^{-3} K^{-4} .

One of the greatest triumphs of quantum theory is that it was able to explain the empirical thermal spectrum of radiation, *yielding the coefficient in terms of fundamental physical constants.*

Next, we will consider a "gas of Fermions". Two important astrophysical examples of Fermion gases are (1) in the early Universe – when all particles existed in thermal equilibrium, including protons, neutrons, electrons, positrons, neutrinos, etc.; and (2) in the interiors of white dwarf stars, where the electrons act as a degenerate Fermion gas.

Just as we did for Bosons, we can compute the average number of Fermions occupying a given state. This is much simpler than for Bosons, however, since each state is occupied by either zero particles (with zero energy) or one particle, with energy E. Thus

$$< N(E) > \ = \ \frac{0 \cdot e^{-0/kT} \ + \ 1 \cdot e^{-E/kT}}{e^{-0/kT} \ + \ e^{-E/kT}} \ = \ \frac{1}{e^{E/kT} + 1}$$

This is known as the *Fermi-Dirac occupation number.* It looks like the Bose-Einstein occupation number, except that in the denominator there is a $+$ rather than a $-$ sign. This shows that at a given energy, (1) <u>the occupation of a Boson state is always greater than that of a Fermion state</u>, and (2) the Fermi occupation number is ≤ 1 .

The occupation numbers we have derived for Bosons and Fermions in fact apply to the case where particles can be freely created and destroyed. This is OK for photons, but for any massive particles (either Bosons or Fermions), the expression as written would apply only at very high temperatures when particle/antiparticle pairs are easily created. Otherwise, we must include the *chemical potential*; including this, the Boltzmann factor $P(E) \propto e^{-E/kT}$ becomes $P(E) \propto e^{-(E-\mu)/kT}$. Making this substitution, $E \rightarrow E - \mu$, the Boson and Fermion mean occupation numbers become

$$< N(E) > \ = \ \frac{1}{e^{(E-\mu)/kT} \ \mp \ 1} \qquad \text{for} \quad \text{Bosons } (-) \text{ and Fermions } (+)$$

At sufficiently high temperature both μ/kT and mc^2/kT are $\ll 1$ ($E^2 - m^2c^4 = c^2p^2$). These limits are appropriate for the early universe. For relativistic particles, $E = cp$. For neutrinos, there are two spin states per particle. In analogy with the photon distributions, we thus have

$$d < n > \ = \ \frac{8\pi}{h^3} \frac{p^2 dp}{e^{cp/kT} + 1} \ = \ \frac{8\pi}{(hc)^3} \frac{E^2 dE}{e^{E/kT} + 1} \quad .$$

For a given neutrino family, the mean number of neutrinos per unit volume in the early universe is

$$< n >= \int d < n >= \frac{8\pi}{(hc)^3} \int_0^\infty \frac{E^2 dE}{e^{E/kT} + 1} = 8\pi \left(\frac{kT}{hc}\right)^3 \int_0^\infty \frac{x^2 dx}{e^x + 1} = 1.80308 \times 8\pi \left(\frac{kT}{hc}\right)^3$$

Note that the value of the dimensionless integral is exactly 3/4 that of the integral for photons we obtained on the previous page (1.80308 vs. 2.40411). Thus, at a given temperature,

$$< n >_{neutrinos} \ = \ \frac{3}{4} < n >_{photons} \quad .$$

Similarly, the neutrino energy density for a given family is

$$u_{neutrino} = \int E \; d < n >= \frac{8\pi}{(hc)^3} \int_0^\infty \frac{E^3 dE}{e^{E/kT}+1} = \frac{8\pi (kT)^4}{(hc)^3} \int_0^\infty \frac{x^3 dx}{e^x + 1} = \frac{8\pi}{(hc)^3} (kT)^4 \times \frac{7}{8} \frac{\pi^4}{15}$$

That is, the energy density in a given neutrino family is exactly 7/8 of the photon energy density:

$$u_{neutrino} \;=\; \frac{7}{8}\, a_{rad} T^4 \quad .$$

Also, since neutrinos are relativistic, the pressure $P_{neutrino} \;=\; \frac{1}{3}\, u_{neutrino}$.

In the early Universe, for every photon there were $\frac{3}{4}$ of a neutrino per family. Assuming 3 neutrino families, the total number density in "relict" neutrinos is therefore $3 \times \frac{3}{4}$ the number density CBR photons. The energy density in relict neutrinos is, similarly, $3 \times \frac{7}{8}$ that of the radiation.

Strictly speaking, correction factors $(T_\nu/T)^3$ and $(T_\nu/T)^4$ must be applied for neutrino number density and energy density, respectively, because the *radiation* temperature increased by a factor $(T/T_\nu) = (11/4)^{1/3}$ after the time of neutrino decoupling, due to the added energy from electron-positron annihilation. This yields a number density of neutrinos of

$$3 \times \frac{3}{4} \times \frac{4}{11} \times n_\gamma \;=\; \frac{9}{11}\, n_\gamma$$

At the present epoch, this works out to ≈ 340 neutrinos cm^{-3}, filling the cosmos. These neutrinos have not, however, been directly detected yet.

(The foregoing notes, aside from minor changes, were written in 2009 for ASTR 320 by Dr. Eve Ostriker.)