① (a) adiabatic means $dQ = 0$.

divide through by C_V and T to obtain

$$\frac{dT}{T} = \left(1 - \frac{C_P}{C_V}\right)\frac{dV}{V} \qquad \text{But } \frac{C_P}{C_V} = \gamma, \text{ so}$$

$$\frac{dT}{T} = (1-\gamma)\frac{dV}{V} \qquad \text{Integrate: } \ln T = (1-\gamma)\ln V + \text{const}$$

exponentiate both sides: $T = \text{const}\ V^{(1-\gamma)}$

Consider two values T_0 & T_1; divide one by the other

$$\left(\frac{T_1}{T_0}\right) = \left(\frac{V_1}{V_0}\right)^{1-\gamma} = \left(\frac{V_0}{V_1}\right)^{\gamma-1}$$

(b) $\gamma = 5/3 \rightarrow \gamma - 1 = 2/3$ We must use the Kelvin temp scale.

$$T_1 = \left(\frac{V_0}{V_1}\right)^{2/3} T_0 \qquad V_1 = \frac{1}{2}V_0 \rightarrow \frac{V_0}{V_1} = 2$$

$$T_1 = 2^{2/3} \cdot 300 = (1.5874)\,300 = \underline{\underline{476.2\ K}} \quad (= 203\ C^\circ)$$

(c) if $\gamma = 7/5$ $\quad \gamma - 1 = \frac{2}{5} = 0.4$

$$T_1^{(d)} = (2)^{0.5}\,300 = (1.3195)\,300 = \underline{\underline{395.85\ K}} \quad (= 123\ C^\circ)$$

(d) $T_1^{(m)} > T_1^{(d)}$. For a monatomic gas, all the energy from the work done by compression goes into kinetic energy of the gas. For a diatomic gas, some of the energy is stored as <u>rotation</u> of the molecules, hence less kinetic energy, hence less temperature increase.

② (a) $$n = \int dn(\vec{p}) = \int \frac{2/h^3 \; d^3p}{e^{cp/kT}-1} = \frac{2}{h^3}\int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta \, d\theta \int_0^{\infty} \frac{p^2 dp}{e^{cp/kT}-1}$$

$$\int_0^{2\pi} d\phi = 2\pi, \quad \int_0^{\pi} \sin\theta \, d\theta = 2 \qquad n = \frac{8\pi}{h^3}\int_0^{\infty} \frac{p^2 dp}{e^{cp/kT}-1}$$

Change variable: let $\frac{cp}{kT} = x$ $\quad p = \frac{kT}{c}x, \; dp = \frac{kT}{c}dx$

Then we have $$n = \frac{8\pi}{h^3}\left(\frac{kT}{c}\right)^3 \int_0^{\infty} \frac{x^2 dx}{e^x - 1}$$

If you search (integral tables, internet..) you find the integral is

$$\int_0^{\infty} \frac{x^2 dx}{e^x-1} = 2\zeta(3) = 2\cdot 1.202057 = 2.40411$$

where $\zeta()$ is the Riemann zeta function.

Thus $$n = \frac{16\pi\,\zeta(3)}{h^3}\left(\frac{k}{c}\right)^3 T^3 = 20.2869 \cdot T^3$$

$$T_1 = 2.73\,K \qquad T_2 = 1400 \cdot 2.73 = 3822\,K$$

so that $n_1 = 412.8 \text{ cm}^{-3}$, $n_2 = 1.13 \cdot 10^{12} \text{ cm}^{-3}$

(b) Here, we just have an additional factor of cp $(|\vec{p}| = p)$. The integrals over ϕ & θ give us 4π as above, so we have

$$u = \frac{8\pi}{h^3} c \int_0^{\infty} \frac{p^3 dp}{e^{cp/kT}-1} = \frac{8\pi c}{h^3}\left(\frac{kT}{c}\right)^4 \int_0^{\infty} \frac{x^3 dx}{e^x - 1}$$

using the same variable change as in part (a).

Now, looking up the integral, $\int_0^{\infty} \frac{x^3 dx}{e^x - 1} = \frac{\pi^4}{15}$,

Thus $u = \left(\frac{8\pi^5}{15}\frac{k^4}{h^3c^3}\right) T^4 = a_{rad} T^4$; $a_{rad} = \frac{8\pi^5 k^4}{15 h^3 c^3}$

Evaluating the constant $a_{rad} = 7.5658 \cdot 10^{-15} \text{ erg cm}^{-3} K^{-4}$

$u(T_1) = \underline{4.202 \cdot 10^{-13}} \text{ erg cm}^{-3}$

$u(T_2) = \underline{1.614} \text{ erg cm}^{-3}$

(3) Setting $K=1$, the equation becomes $(\dot{R})^2 = 2\frac{A}{R} - c^2$

Check if $t = \frac{A}{c^3}(x - \sin x)$ and $R = \frac{A}{c^2}(1-\cos x)$ is a solution.

$$\dot{R} = \frac{dR}{dt} = \frac{dR/dx}{dt/dx} = \frac{\frac{A}{c^2}\sin x}{\frac{A}{c^3}(1-\cos x)} = c\frac{\sin x}{1-\cos x}$$

So the left hand side is

$$(\dot{R})^2 = c^2\frac{\sin^2 x}{(1-\cos x)^2} = c^2\frac{1-\cos^2 x}{(1-\cos x)^2} = c^2\frac{(1+\cos x)\cancel{(1-\cos x)}}{(1-\cos x)\cancel{(1-\cos x)}}$$

or $$\boxed{(\dot{R})^2 = c^2\frac{1+\cos x}{1-\cos x}}$$

Now look at the right hand side:

$$2\frac{A}{R} - c^2 = 2\frac{A}{\frac{A}{c^2}(1-\cos x)} - c^2 = c^2\left[\frac{2}{1-\cos x} - 1\right]$$

write $1 = \frac{1-\cos x}{1-\cos x}$

$$2\frac{A}{R} - c^2 = c^2\left[\frac{2-(1-\cos x)}{1-\cos x}\right] = c^2\frac{1+\cos x}{1-\cos x}$$

which agrees with the result for $(\dot{R})^2$ above. QED,