

## 1. Hydrostatic Equilibrium and Stellar Structure

### 1.1. The Isothermal Sphere and Stellar Cores

We previously discussed truncated isothermal spheres and their stability as applied to clouds in the interstellar medium ("Bonnor-Ebert spheres"). A similar question of stability arises in the case of stellar evolution. When hydrogen-burning has exhausted the supply of hydrogen in the core of a star, an inert helium core remains. This core has no energy source, and hence will become isothermal – the temperature will be constant at the value set by the inner edge of the hydrogen-burning shell surrounding this core. This core will have the structure of a *truncated isothermal sphere*.

Students sometimes suppose that isothermal regions in stars will have constant density, but this is not the case. The density must increase toward the center to satisfy the equation of hydrostatic equilibrium. While the star is burning hydrogen in its core, the temperature is highest at the center. For an ideal gas, this temperature gradient helps support the star, while for an isothermal core, the density gradient alone must support the stellar mass, and the density gradient steepens.

As hydrogen burning in the surrounding shell adds mass to the inert helium core, the isothermal sphere extends further from the center and the center/edge density ratio increases. Just as in the case of the Bonnor-Ebert spheres, if we consider a fixed core mass  $M_c$ , and decrease the core radius  $R_c$ , we find that the pressure at  $R_c$  reaches a maximum  $P_{max}$  and then decreases – beyond that point the core is unstable. Thus a solution is possible only if the pressure of the envelope is below that maximum pressure. We further find that  $P_{max}$  decreases with increasing core mass as  $M_c^{-2}$ . This means that as an isothermal core grows, it will reach a point of instability – the core will rapidly contract and heat, and this may lead to the onset of helium burning. This limit to the ratio of the core mass to the total stellar mass is known as the *Schönberg-Chandrasekhar limit* (sometimes *Chandrasekhar-Schönberg limit*). It is approximately

$$\frac{M_c}{M} \leq 0.37 \left( \frac{\mu_{env}}{\mu_{core}} \right)^2$$

where  $\mu_{env}$  is the mean molecular weight of the envelope ( $\simeq 0.66$ ), and  $\mu_{core}$  that of the core ( $\simeq 1.33$ ), so that  $M_c/M \leq 0.09$ .

Note: This is not the same as the *Chandrasekhar mass*, the mass limit for a white dwarf. That is for material which is degenerate, while the stellar core discussed here is of lower density and behaves as an ideal monatomic gas.

## 1.2. A Lower Limit to the Central Pressure in Stars

Let us recall the equation of hydrostatic equilibrium and the equation for  $M_r$  :

$$\frac{dP}{dr} = -\frac{GM_r}{r^2} \rho \quad \text{and} \quad \frac{dM_r}{dr} = 4\pi r^2 \rho$$

Let's combine these equations

$$\frac{dP}{dM_r} = \frac{dP}{dr} \frac{dr}{dM_r} = \left( -\frac{GM_r}{r^2} \rho \right) \left( \frac{1}{4\pi r^2 \rho} \right)$$

to obtain the equation of hydrostatic equilibrium in terms of the variable  $M_r$ :

$$\boxed{\frac{dP}{dM_r} = -\frac{GM_r}{4\pi r^4}}$$

Now integrate this expression from  $r = 0$  to  $r = R$ :

$$\int_{P(0)}^{P(R)} dP = -\frac{G}{4\pi} \int_0^M \frac{M_r}{r^4} dM_r$$

The pressure at the surface,  $P(R)$ , is zero so

$$\int_{P(0)}^{P(R)} dP = P(R) - P(0) = -P_c$$

Now, since  $1/r^4 \geq 1/R^4$ , we have the inequality

$$P_c > \frac{G}{4\pi R^4} \int_0^M M_r dM_r$$

and thus

$$P_c > \frac{G M^2}{8\pi R^4} = 4.49 \times 10^{14} \left( \frac{M}{M_\odot} \right)^2 \left( \frac{R_\odot}{R} \right)^4 \text{ dyne cm}^{-2}$$

This is not a very close limit (it's orders of magnitude too small), but it does show that the central pressures of stars must be quite high. We can actually get a better estimate by assuming that the density through the star is constant. That assumption leads to

$$P_c = \frac{3GM^2}{8\pi R^4}$$

(You might try proving this.) That this is a *lower limit* requires demonstrating the (reasonable) proposition that a star where the density is not constant, but increases inwards, must have a higher central pressure.

The mean density of a star is  $\langle \rho \rangle = M/V = 3M/4\pi R^3$ , which for the Sun works out to  $\langle \rho \rangle_\odot = 1.41 \text{ gm cm}^{-3}$ . If we write the central pressure of our constant density star as

$$P_c = \frac{3GM^2}{8\pi R^4} = \frac{G M}{2 R} \left( \frac{3M}{4\pi R^3} \right) = \frac{G M}{2 R} \rho$$

then we can equate this to the pressure from the ideal gas law,

$$\frac{G M}{2 R} \rho = P_c = \frac{N_A k}{k} \rho T_c ,$$

cancel the  $\rho$ , and solve for  $T_c$  to obtain

$$T_c = \frac{G \mu M}{2 N_A k R} = 11.5 \times 10^6 \mu \left( \frac{M}{M_\odot} \right) \left( \frac{R_\odot}{R} \right) \text{ K}$$

With  $\mu = 0.66$ , this gives us  $T_c = 7.6 \times 10^6$  K for the sun; the actual value is almost twice as high ( $T_c = 14.4 \times 10^6$  K). But remember that the constant density model only gives us a lower limit.

### 1.3. The Virial Theorem for Stars

The volume inside radius  $r$  is just  $V_r = \frac{4}{3}\pi r^3$ . Let us multiply both sides of the  $dP/dM_r$  version of the hydrostatic equilibrium equation by  $V_r$  to obtain

$$V_r dP = -\frac{G}{3} \frac{M_r dM_r}{r}$$

and integrate this over the whole star

$$\int_{P(0)}^{P(R)} V_r dP = -\frac{1}{3} \int_0^M \frac{GM_r}{r} dM_r$$

Now, it turns out that the integral on the right hand side is 1/3 the gravitational potential energy of the star. We can see this if we consider the work against gravity we need to expend if we disperse the star by lifting off successive layers of mass  $dM_r$ . The work required to lift one layer  $dM_r$  from  $r$  to  $\infty$  is

$$dW = dM_r \cdot \text{force} \cdot \text{distance} = dM_r \cdot \int_r^\infty \frac{GM_r}{r'^2} dr' = dM_r \cdot \left[ -\frac{GM_r}{r'} \right]_r^\infty = dM_r \cdot \frac{GM_r}{r}$$

and this must be integrated over all the mass shells to obtain the energy needed to disperse the star. Thus the negative of this quantity is the gravitational potential energy  $\Omega$ :

$$\Omega = - \int_0^M \frac{GM_r}{r} dM_r$$

and the equation above becomes

$$\int_{P(0)}^{P(R)} V_r dP = \frac{1}{3} \Omega .$$

On the left hand side of this equation we employ integration by parts.

$$\int V_r dP = [P V_r]_0^R - \int P dV_r$$

Since the pressure is zero at  $r = R$  and the volume  $V_r$  is zero at  $r = 0$ , the first term on the RHS vanishes and we are left with

$$\int_0^R P dV_r = -\frac{1}{3} \Omega .$$

Now from our earlier discussion of the thermal properties of gases, we recall that the pressure is related to the internal energy by  $P = (\gamma - 1) u$ . Therefore

$$\int_0^R P dV_r = (\gamma - 1) \int_0^R u dV = (\gamma - 1) U ,$$

where  $U$  is the total internal (thermal) energy of the star. Thus we arrive at

The Virial Theorem:  $\Omega = -3 (\gamma - 1) U$

Consider the case relevant to most stars, where the material behaves as an ideal, monatomic gas. Then  $\gamma = 5/3$ ,  $(\gamma - 1) = 2/3$  and the virial theorem becomes

$$\Omega = -2 U \quad \text{or} \quad U = -\frac{1}{2} \Omega .$$

It is also informative to look at the total (thermal + gravitational) energy  $E_{total}$  :

$$E_{total} = \Omega + U = \frac{1}{2} \Omega = -U .$$

It is no surprise that  $E_{total} < 0$ , since stars are gravitationally bound. By contrast, consider the case where  $\gamma = 4/3$ . (Recall that this is the case for radiation. There are other examples as well, such as relativistic degenerate material.) Then

$$\gamma = \frac{4}{3} \implies (\gamma - 1) = \frac{1}{3} \implies \Omega = -U \implies E_{total} = 0 !$$

It's not just that we can't make a gravitationally bound object out of radiation alone; this also implies that very massive stars, where most of the pressure is due to radiation not gas, are only weakly bound, as  $E_{total}$  will approach zero.

So how can we estimate the gravitational potential energy  $\Omega$ ? Let us start again with our constant density approximation. It is easy to see that in this case  $(M_r/M)$  is proportional to the ratio of volumes:

$$M_r = \left( \frac{r^3}{R^3} \right) M \quad \text{for } \rho = \text{const.}$$

Then

$$dM_r = 3M \left( \frac{r^2}{R^3} \right) dr \implies \Omega = - \int_0^R \frac{G}{r} M \left( \frac{r^3}{R^3} \right) 3M \frac{r^2}{R^3} dr$$

$$\Omega = - \frac{3GM^2}{R^6} \int_0^R r^4 dr = - \frac{3GM^2}{R^6} \left[ \frac{1}{5} r^5 \right]_0^R = - \frac{3}{5} \frac{GM^2}{R}$$

So we can write our result as

$$\Omega = -q \frac{GM^2}{R} \quad \text{where, for } \rho = \text{constant, } q = \frac{3}{5}$$

It turns out that the form given above is quite general, with the value of  $q$  depending on the distribution of matter within the star. The *minimum* value for  $q$  is the constant density value of  $3/5$ ; a more typical value might be  $q \simeq 1.5$ . For the density distribution of a polytrope (discussed in the next section) we have the remarkable result that  $q = 3/(5-n)$ , where  $n$  is the index of the polytrope.

To look at numerical values, we can introduce the solar mass and radius:

$$\Omega = -3.8 \times 10^{48} q \left( \frac{M}{M_\odot} \right)^2 \left( \frac{R}{R_\odot} \right)^{-1} \text{ erg}$$

When a star of mass  $M$  and radius  $R$  is formed,  $-\Omega$  is the amount of gravitational energy released; half of this energy goes into thermal energy that the star needs to support itself, while the other half must be radiated away. (Well, not quite half: some energy is used dissociating  $H_2$  molecules and then ionizing the gas.)

Going back to the virial theorem,  $U = -\frac{1}{2}\Omega$ , we see we have an estimate for the average thermal energy of the Sun:  $U \sim 2 \times 10^{48}$  erg. Dividing by the Sun's volume,  $V_\odot = \frac{4}{3}\pi R_\odot^3 = 1.41 \times 10^{33}$  cm<sup>3</sup>, the energy per unit volume is  $\langle u_\odot \rangle = U/V_\odot = 1.4 \times 10^{15}$  erg cm<sup>-3</sup>. Now the thermal energy is related to the temperature by  $\langle u_\odot \rangle = \frac{3}{2}nk \langle T_\odot \rangle$ . The number of particles is  $n = N_A \rho / \mu$ , so can write  $\langle T_\odot \rangle = \frac{2\mu}{3N_A \rho k} \langle u_\odot \rangle$ . Plugging in the solar values, we find  $\langle T_\odot \rangle \simeq 5 \times 10^6$  K. So without much physics beyond the virial theorem we are able to establish that the interior temperature of the Sun is millions of degrees.

The virial theorem has many uses which we cannot go into here, but it is worth noting that it can be applied to *part* of a star: in our derivation we just integrate out to some radius  $r = R_s$  rather than the surface  $r = R$ . Then we find that

$$\Omega + 3(\gamma - 1)U = 3P_s V_s \text{ ,}$$

where  $\Omega$  and  $U$  refer to the part of the star within radius  $R_s$ ,  $P_s$  is the pressure at that radius and  $V_s$  the enclosed volume.

#### 1.4. Polytopic Stellar Models

A proper understanding of stellar structure draws on many different areas of physics, e.g., thermonuclear reactions, the atomic physics of stellar opacity, etc. However, we can go a good bit further using the principles of hydrostatic equilibrium. There is a rather beautiful mathematical theory developed over a century ago, the theory of polytropic models. Our discussion of the isothermal sphere was a special case of such models. Recall equation (26) that we derived there:

$$\frac{1}{r^2} \frac{d}{dr} \left[ \frac{r^2}{\rho} \frac{dP}{dr} \right] = -4\pi G \rho .$$

In general,  $P = P(\rho, T)$  and the equation above must be coupled with an equation for  $T$ . But there are cases where we may assume that  $P = P(\rho)$  only. This is the case with the so-called polytropes, which are the solutions which follow when  $P \propto \rho^\gamma$ , where  $\gamma$  is a constant. (In some – but not all – cases  $\gamma$  is the ratio of specific heats we have used before.) It is usual to define a *polytropic index*  $n$ , which is related to gamma by  $\gamma = 1 + (1/n)$ . Then the pressure can be written

$$P = K \rho^{1+\frac{1}{n}}$$

where  $K$  is another constant. We now replace  $\rho$  by a new variable  $\theta$  defined by the relation

$$\rho = \rho_c \theta^n ,$$

where  $\rho_c$  is the density at the center of the star. The pressure then becomes

$$P = K \rho_c^{\frac{n+1}{n}} \theta^{n+1} .$$

We also introduce a dimensionless variable  $\xi$  in place of the radius  $r$  by  $r = a\xi$ . With the right choice of  $a$  (see, e.g. Choudhuri, p 133), our equation becomes

$$\frac{1}{\xi^2} \frac{d}{d\xi} \left( \xi^2 \frac{d\theta}{d\xi} \right) = -\theta^n .$$

This is known as the **Lane-Emden equation**. It is a second-order differential equation, and needs two boundary conditions to define a solution. These conditions are  $\theta = 1$  and  $d\theta/d\xi = 0$  at the center  $\xi = 0$ . If  $n = 0, 1$  or  $5$ , there are analytic solutions: for example, for  $n = 1$ ,  $\theta(\xi) = (\sin \xi)/\xi$  is a solution. Otherwise, the equation must be integrated numerically. As with the case of the isothermal sphere (which, by the way, can be considered the  $n = \infty$  case), the numerical integration cannot be started at  $\xi = 0$  because of the  $1/\xi^2$  term. Again, we use a series expansion near the origin:

$$\theta(\xi) = 1 - \frac{1}{6} \xi^2 + \frac{n}{120} \xi^4 - \frac{n(8n-5)}{15120} \xi^6 + \dots$$

As the index  $n$  varies, we obtain a series of models with increasing central mass concentration (the  $n = 0$  case is in fact the constant density model). Here we show some solutions normalized to the same radius and central density.

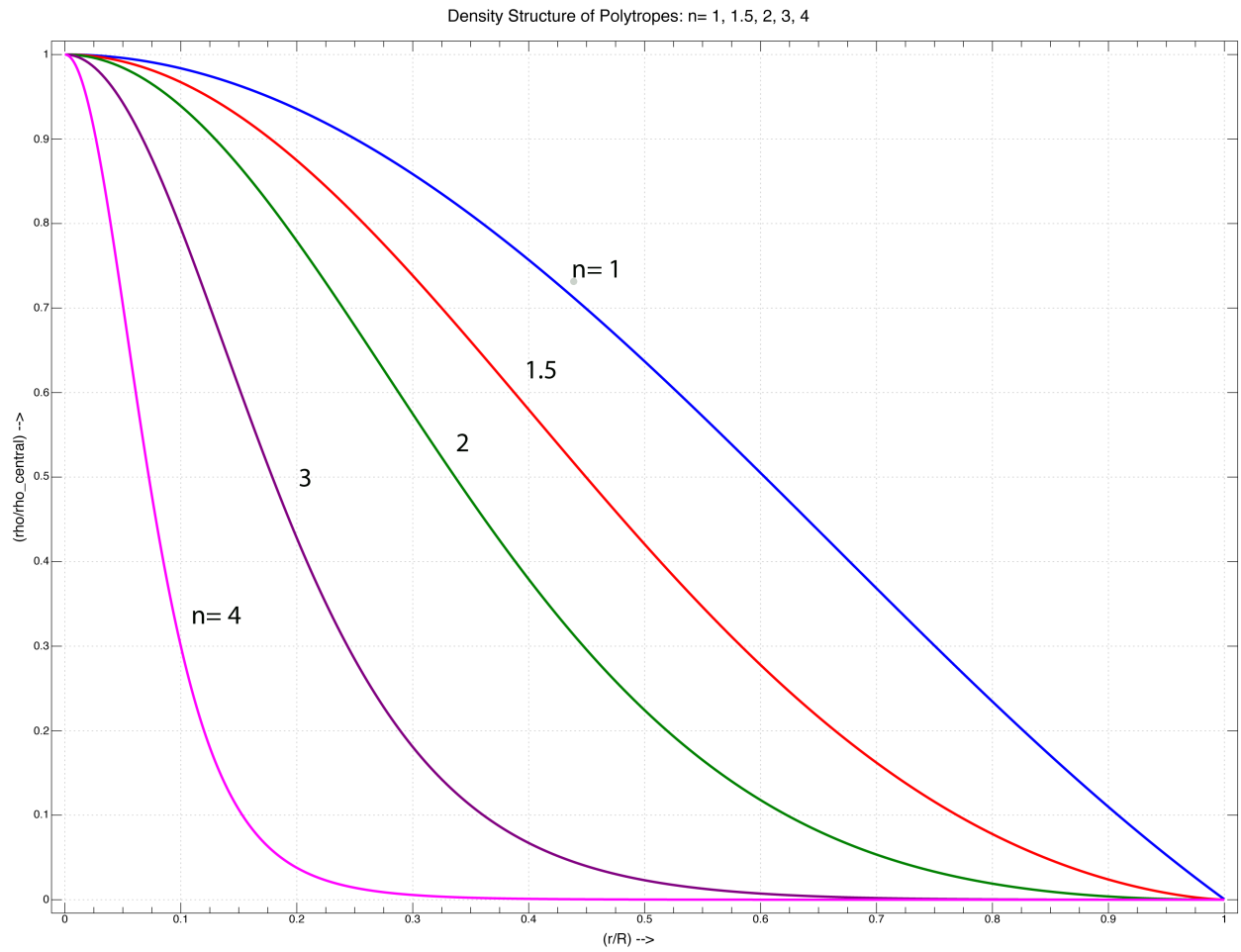


Fig. 1.— Density Distribution as a Function of Polytropic Index

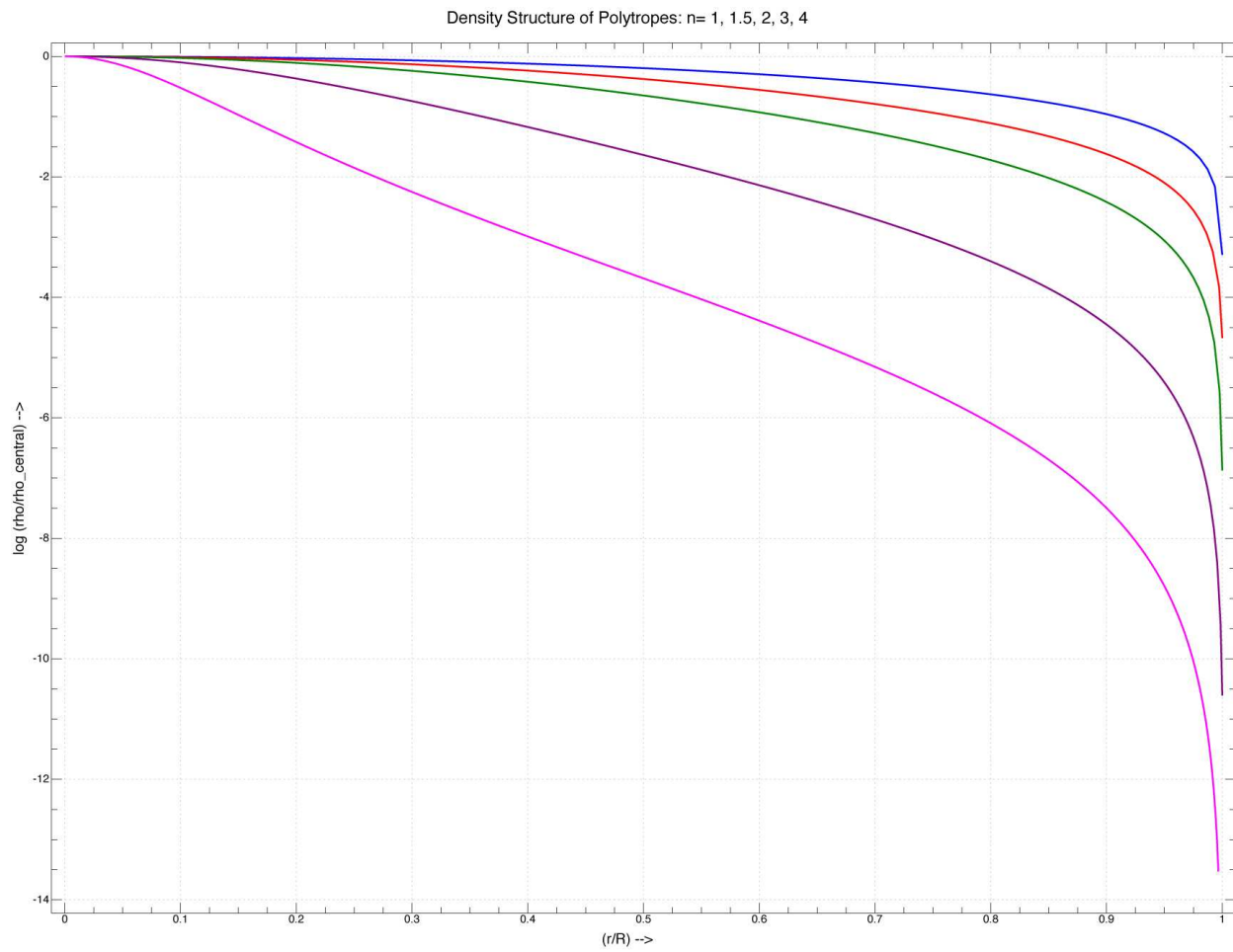


Fig. 2.— Log Density Distribution as a Function of Polytropic Index



Before astronomers learned to make “real” stellar models, polytropes were quite important. As shown in the next figure, where I have plotted the actual density distribution of the Sun compared with an  $n = 3$  polytrope, there is a real similarity. While polytropes are only of historical interest for the study of main sequence and giant stars, they still have their use. For example, white dwarf stars of low mass (non-relativistic) look like polytropes of index  $n = 1.5$  ( $\gamma = 5/3$ ), while with increasing density, as the electron degeneracy becomes relativistic, the structure of a white dwarf approaches that of an  $n = 3$  ( $\gamma = 4/3$ ) polytrope.

### 1.5. The Equations of Stellar Structure

The proper construction of a stellar model requires the simultaneous solution of four differential equations. These basic equations are (Maoz, p 42; Choudhuri, p 71):

$$\begin{aligned} \frac{dP}{dr} &= - \frac{GM_r}{r^2} \rho \\ \frac{dM_r}{dr} &= 4\pi r^2 \rho \\ \frac{dT}{dr} &= - \frac{3L_r \kappa \rho}{4\pi r^2 4ac T^3} \\ \frac{dL_r}{dr} &= 4\pi r^2 \rho \epsilon \end{aligned}$$

In addition, we need what is called *the equation of state*. This is the relation between pressure, temperature and density. In stars like the sun, this would be the ideal gas law:

$$P = \frac{N_A k}{\mu} \rho T \ ,$$

while for massive stars we have to add a term  $\propto T^4$  due to radiation pressure. The equation of state becomes even more complicated for the cores of stars in the late stages of stellar evolution, when the quantum effects known as “degeneracy” set in.

We are familiar with the first two of these four equations. The third equation for the temperature gradient  $dT/dr$  has a lot of physics behind the quantity  $\kappa$ , which represents the mean opacity of the stellar material. It is a complex function of temperature, density and chemical composition.

The fourth equation tells us how the luminosity  $L_r$  flowing outward increases due to energy generation by nucleosynthesis. The symbol  $\epsilon$  represents the energy generation per gram, and is an extremely strong function of temperature.

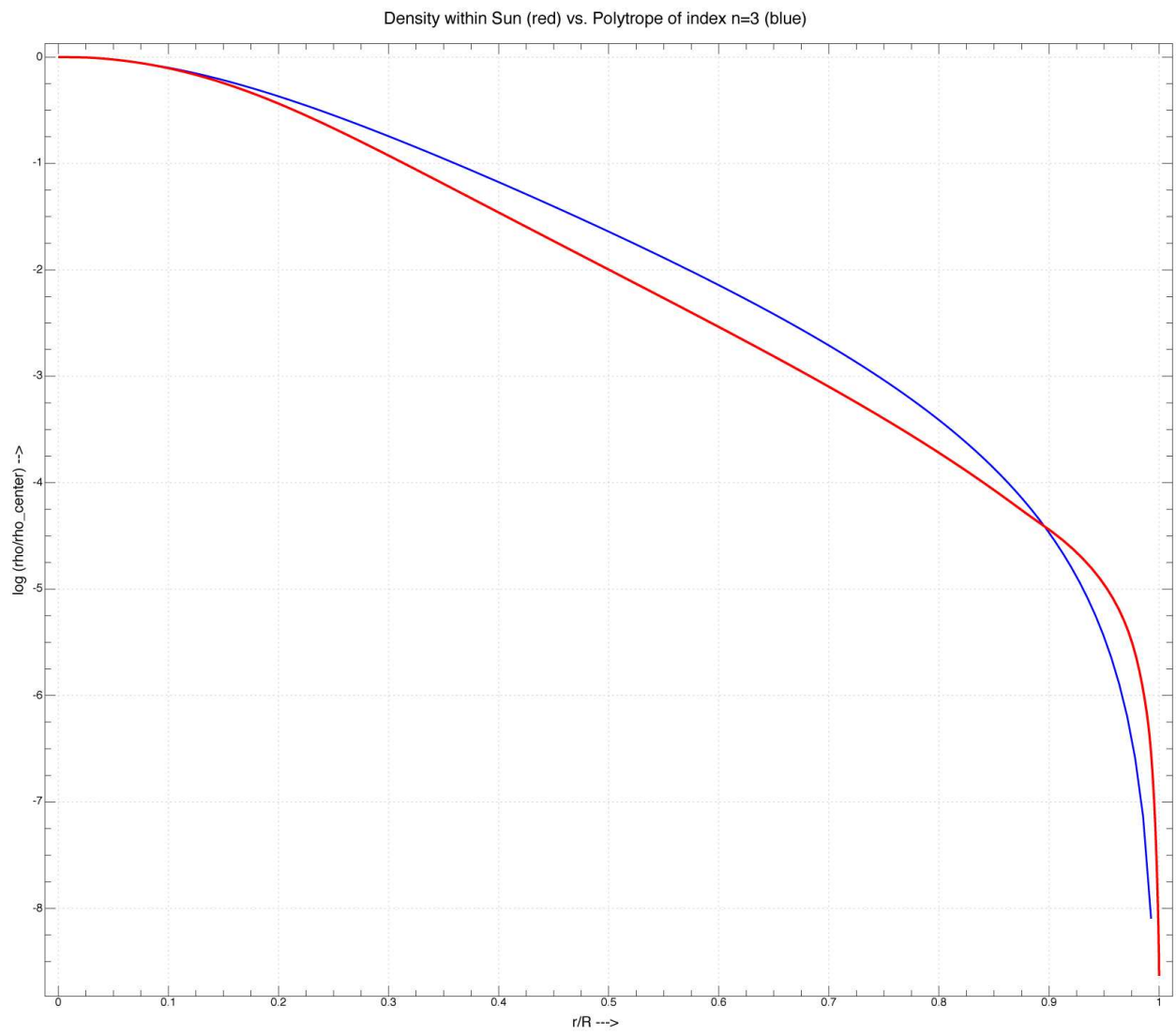


Fig. 3.— The Solar Density Distribution Compared to an  $n=3$  Polytrope