1. White Dwarfs and Neutron Stars

We now want to see the implications of degeneracy for stellar structure. But first, it is useful to draw back and get an overview of the conditions we might encounter inside stars and where different "equations of state" (i.e., the $P = P(\rho, T)$ relationship) will apply.

1.1. Equations of State in the T-ρ Plane

We can find an estimate of where we must switch from ordinary gas pressure to degenerate electron pressure by equating the two expressions to see at what ρ and T they will be equal:

$$P_{nr} \;=\; \left(\frac{3}{\pi}\right)^{2/3} \frac{h^2}{20\; m} \left(\frac{N_A}{\mu_e}\right)^{5/3} \rho^{5/3} \;=\; \frac{N_A}{\mu_e} k\; \rho\; T \;=\; P_{ideal} \tag{1}$$

after some simplification, we find

$$\rho \;=\; \left(\frac{3}{\pi}\right) \left(\frac{20\; k\; m}{h^2}\right)^{3/2} \left(\frac{\mu_e}{N_A}\right) T^{3/2} \;=\; 2.38 \times 10^{-8}\; \mu_e\; T^{3/2} \tag{2}$$

Thus for example, if $T = 10^7$ K and $\mu_e = 2$, degeneracy sets in for $\rho > 1500$ g cm^{-3}, while if $T = 10^5$ K, it sets in at 1.5 g cm^{-3}. The degeneracy will become relativistic for densities above $x_F = 1$, which is $\rho = 1.07 \times 10^6\; \mu_e$ g cm^{-3}, independent of T. Another equation of state is important in very massive stars, where the temperature is high but the density low. Then radiation pressure becomes important. Since $P_{rad} = \frac{1}{3} a_{rad}\, T^4$, at sufficiently high temperatures it will dominate gas pressure.

The figure on the next page shows what equation of state applies for various values of temperature and density. The red dot shows the sun's core, which is still in the ideal gas region, even though the density is over 100 g cm^{-3}.

1.2. The Structure of White Dwarfs

We saw that the pressure of degenerate electrons has two limiting cases, both of the form $P \propto \rho^\gamma$, where in the non-relativistic limit $\gamma = 5/3$, while in the highly relativistic limit $\gamma = 4/3$. These are examples of the polytropic equation of state, where the polytropic index n is related to γ by $\gamma = 1 + 1/n$. Thus a white dwarf with non-relativistic electrons will have the structure of a polytrope of index $n = 1.5$, while if the electrons are relativistic, the structure will be that of a $n = 3$ polytrope. Go back to the class notes on "Stellar Structure" and look at the density distributions for the various polytropes: Evidently, the structure of a white dwarf will be between the curves for $n = 1.5$ and $n = 3$.

Clearly, the interior of a white dwarf shows a great range of density. In our discussion of polytropes in the notes on "Stellar Structure", we introduced a variable θ such that

Regions of the log T - log ρ plane with different equations-of-state.

(Red dot indicates center of present-day sun.)

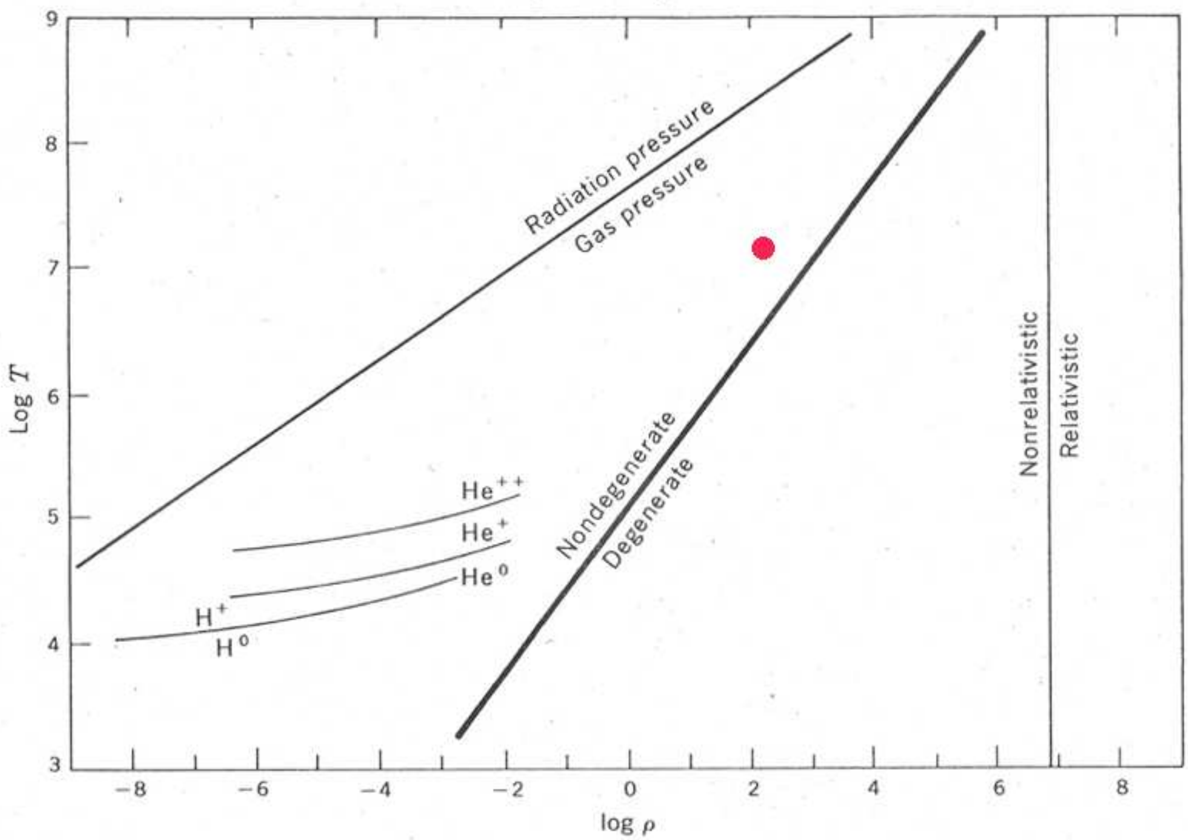


Fig. 2-11 Zones of the equation of state of a gas in thermodynamic equilibrium. Radiation pressure dominates the gas pressure in the upper left-hand corner. The remaining boundaries are similar to those in Fig. 2-7. Also included for comparison are the transition strips in a hydrogen-dominated gas between H^0 and H^+, between He^0 and He^+, and between He^+ and He^{++}.

$\rho(r) = \rho_c\, \theta^n(r)$, and obtained the Lane-Emden (polytrope) equation:

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right) \;=\; -\,\theta^n \; . \tag{3}$$

This equation must be solved by numerical integration. One of the results of the solutions is the relation between the central density, ρ_c and the mean density $\overline{\rho}$. We find that for the non-relativistic case ($n = 1.5$) that $\rho_c = 6.0\,\overline{\rho}$ while for the relativistic case ($n = 3$) the central condensation is greater: $\rho_c = 54\,\overline{\rho}$. Most white dwarfs, however, will fall between these extreme cases. And such an intermediate case is *not* a polytrope with some $1.5 < n < 3$. Rather, we must use the more general expression for degenerate pressure given by equation (17) of our notes on "Degenerate Pressure". In 1935 Chandrasekhar showed that use of this equation leads to the following differential equation:

$$\frac{1}{\eta^2}\frac{d}{d\eta}\left[\eta^2\frac{d\phi}{d\eta}\right] \;=\; -\left[\phi^2 \;-\; \frac{1}{y_0^2}\right]^{3/2} \tag{4}$$

We see that it closely resembles the polytrope equation (3). The parameter y_0 determines what mass white dwarf the model represents, with the more massive models being more relativistic in their central regions. In the limit as $y_0 \rightarrow \infty$ so that $1/y_0^2 \rightarrow 0$, the equation becomes, in fact, the polytrope equation for $n = 3$. The other limit is $y_0 \rightarrow 1$, which, though it's not so obvious, gives the $n = 1.5$ polytrope. The intermediate values, which represent real white dwarfs, are shown in the figure on the next page.

When we examine the mass and radius of the various solutions, we find a strange thing. The larger the mass of the star, the smaller its radius! In fact, for the lower mass white dwarfs, we find $R \propto M^{-1/3}$. This behavior can be derived from the virial theorem, making use of the $P \propto \rho^{5/3}$ equation of state, as you are asked to do in the homework. (The virial theorem approach will not, of course, give an accurate value for the constant of proportionality.) However, as the mass increases to the point where much of the star is becoming relativistic, an even stranger thing occurs: the radius begins to decrease toward zero. On page 5, we show the calculated mass-radius relations.

We may well ask what is happening here. It turns out that the slightly softer relativistic equation of state is not able to halt the contraction of the star against gravity. We can see this from the virial theorem. The virial theorem is

$$3(\gamma - 1)\; U \;=\; -\,\Omega \qquad \text{where } U \text{ is the internal energy and the} \tag{5}$$

gravitational potential energy Ω can be written, for a polytrope of index n , as follows:

$$\Omega \;=\; -\,\frac{3}{5-n}\;\frac{GM^2}{R} \tag{6}$$

For the relativistic case, $\gamma = 4/3$ and $n = 3$, so we have

$$U \;=\; \frac{3}{2}\,\frac{GM^2}{R} \tag{7}$$

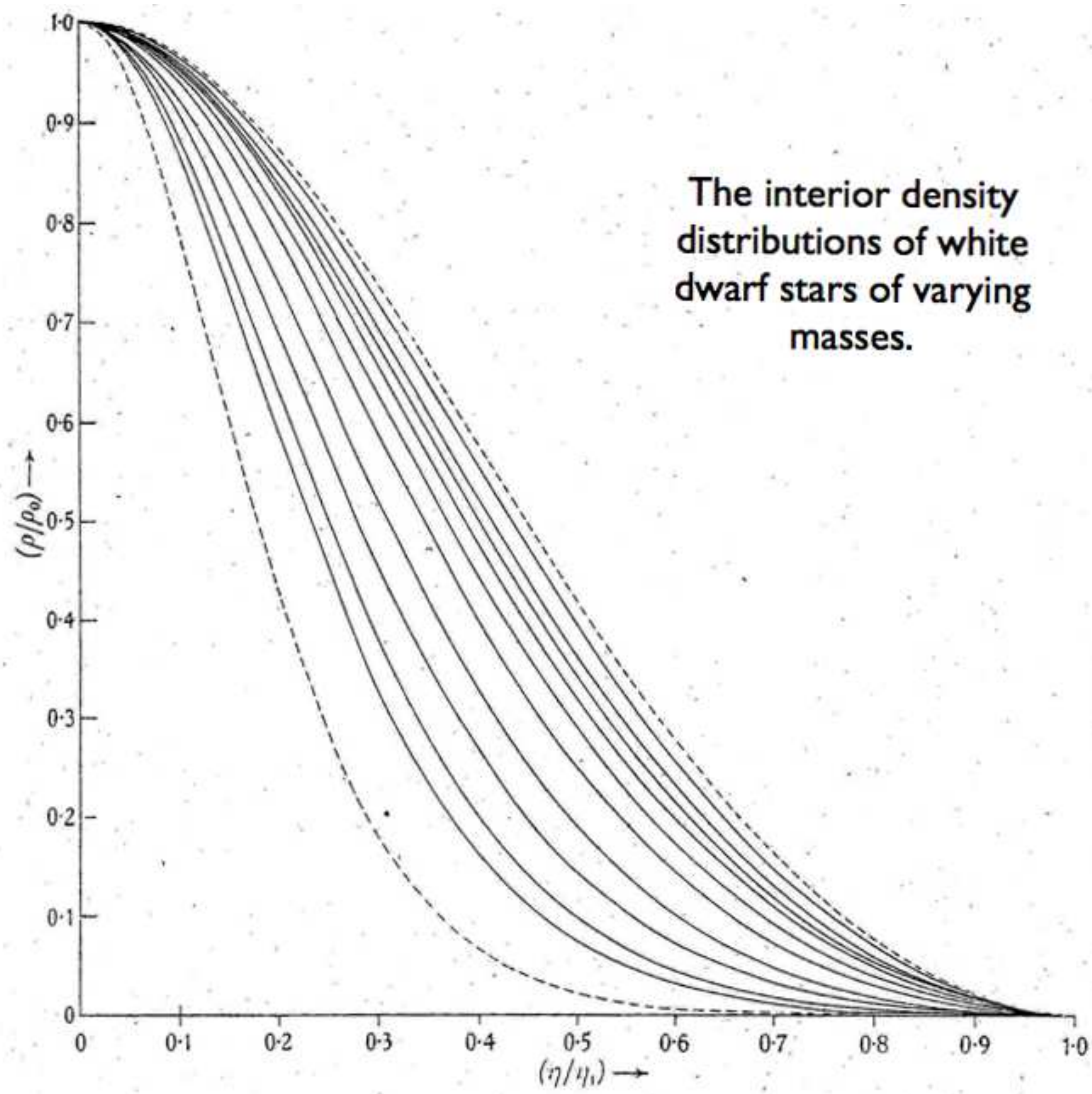


FIG. 33.—The relative density distributions in the completely degenerate configurations. The upper dotted curve corresponds to the polytropic distribution $n = \frac{3}{2}$, and the lower dotted curve to the polytropic distribution $n = 3$. The inner curves represent the density distributions for $1/y_0^2 = 0.8$, 0.6, 0.5, 0.4, 0.3, 0.2, 0.1, 0.05, 0.02, and 0.01, respectively.

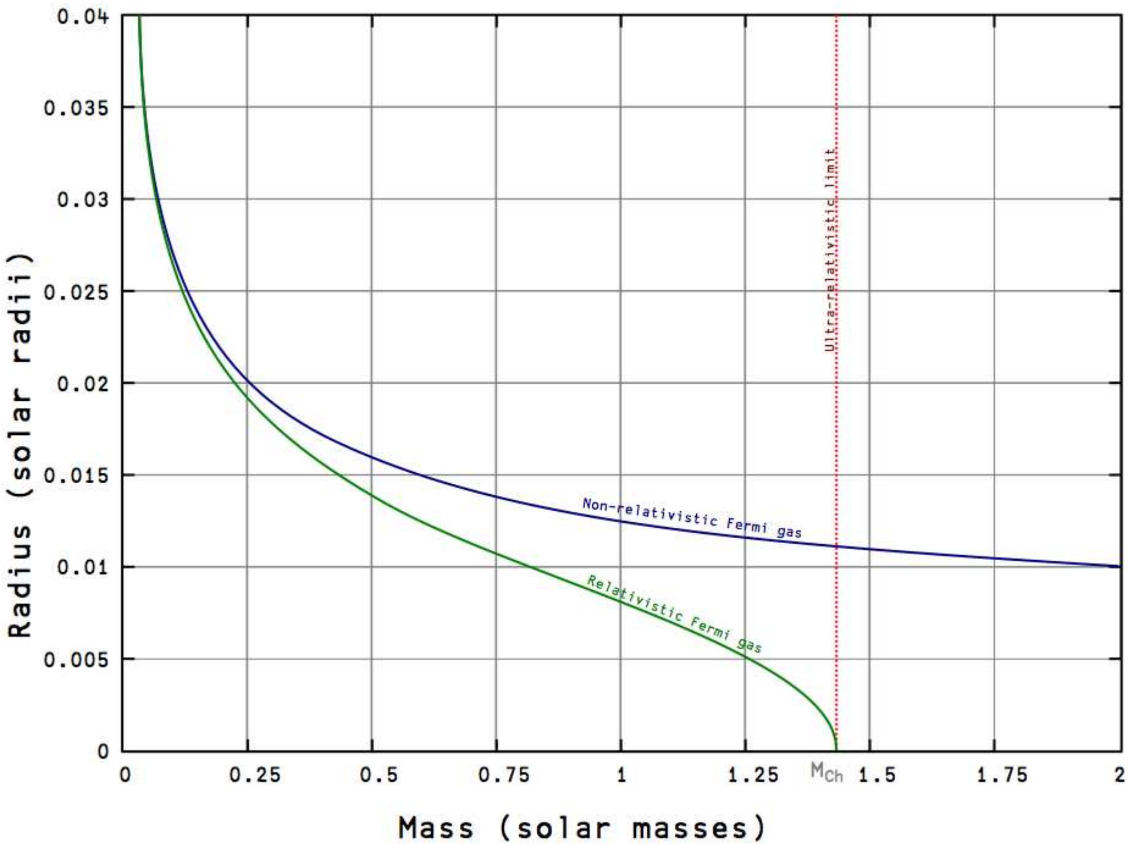


Fig. 1.— The white dwarf mass-radius relation. The green line uses the correct equation of state.

The volume of the star is $V = \frac{4}{3}\pi\ R^3$, and the total internal energy $U = Vu$, where u is the internal energy per unit volume. But u is related to the pressure: $u = P/(\gamma - 1) = 3P$. So we have

$$4\pi R^3 \cdot P \;=\; \frac{3}{2}\,\frac{GM^2}{R} \quad . \tag{8}$$

For the pressure, we insert the relativistic form we obtained earlier:

$$P \;=\; \frac{hc}{8}\,\left(\frac{3}{\pi}\right)^{1/3}\left(\frac{N_A}{\mu_e}\right)^{4/3}\rho^{4/3} \quad . \tag{9}$$

which leads to

$$4\pi R^3\,\frac{hc}{8}\,\left(\frac{3}{\pi}\right)^{1/3}\left(\frac{N_A}{\mu_e}\right)^{4/3}\rho^{4/3} \;=\; \frac{3}{2}\,\frac{GM^2}{R} \quad . \tag{10}$$

Next, we make a key approximation, which is dimensionally correct, but will reduce the accuracy of our result. We need to use some average value for the density, and we will simply use the average over the whole star: $<\rho>= M/V = 3M/4\pi R^3$. Then equation (10) becomes

$$8\pi R^3\,\frac{hc}{8}\,\left(\frac{3}{\pi}\right)^{1/3}\left(\frac{N_A}{\mu_e}\right)^{4/3}\left(\frac{3M}{4\pi R^3}\right)^{4/3} \;=\; 3\,\frac{GM^2}{R} \quad . \tag{11}$$

After some cancellation we have

$$\frac{hc}{4G}\,\left(\frac{3}{\pi}\right)^{1/3}\left(\frac{N_A}{\mu_e}\right)^{4/3}\left(\frac{3}{4\pi}\right)^{1/3}\frac{M^{1/3}}{R} \;=\; \frac{M}{R} \quad . \tag{12}$$

And here is the startling result: the star's radius R cancels out, leaving an equation in M only! This means that no adjustment of the radius can bring the star into agreement with the virial theorem. Solving this equation for the mass gives us

$$M \;=\; \left(\frac{hc}{4G}\right)^{3/2}\left(\frac{3}{2\pi}\right)\left(\frac{N_A}{\mu_e}\right)^2 \;=\; 8.8\times 10^{32}\mathrm{g} \;=\; 0.44\ M_\odot \quad , \tag{13}$$

where we have assumed $\mu_e = 2$. So we see that there is a critical value for the mass of a white dwarf – even though the numerical value we obtain is too small by a factor of three. (Indeed, we would expect our value to be too small, since we used a straight average of the density, while the pressure depends upon $\rho^{4/3}$ and the central density is ~ 50 times the average density.)

To get the correct numerical value, we must appeal to the actual solution of the polytropic equation for $n = 3$ (e.g., Choudhuri, pp 135-136). The famous result is

$$M_{Ch} \;=\; 1.456\,\left(\frac{2}{\mu_e}\right)^2 M_\odot \qquad \text{The } \textit{Chandrasekhar Mass} \tag{14}$$

It is hard to overstate the importance of this result. If a star, during its lifetime, cannot loose enough mass to slip below M_{Ch} , it is destined to end as a supernova. (Stars apparently try hard to loose their mass – even stars that start out at $5M_\odot$ manage to expel enough mass to end as white dwarfs.)

1.3. White Dwarf Temperatures and Their Evolution

White dwarfs get their name from fact that many of them have high surface temperatures (e.g., 16,000K for 40 Eridani B, the first white dwarf discovered) and thus appear white in color. But their surface temperatures are low compared to the temperatures of the *interiors* of these stars, which may be millions or tens of millions K.

When we speak of the interior temperature of a white dwarf, we are referring to the ions (nuclei of He, C, O, etc.) which are not degenerate and have a Maxwellian velocity distribution. The degenerate electrons have kinetic energies much higher than the ions, but they cannot give up their energy since the Pauli exclusion principle prevents them from moving to lower (filled) energy levels. However, the streaming electrons do result in a uniform temperature throughout the interior, by providing efficient conduction – the same exclusion principle prevents them from having energy-changing collisions.

The hot interior is insulated by a thin layer of non-degenerate gas near the surface. This non-degenerate layer is less than a percent of the radius of the white dwarf but in this layer the temperature drops from $\sim 10^7$ K to the surface temperature of $\sim 10^4$ K. Radiation from the surface slowly drains the ionic thermal energy of the interior, but this will take billions of years. The luminosity of the white dwarf is

$$L = 4\pi R^2 \sigma T_{eff}^4 \tag{15}$$

where T_{eff} is the effective temperature of the surface. As a white dwarf cools, its radius R does not change. Thus, in the H-R diagram, it cools along a track with $L \propto T_{eff}^4$. As the temperature drops by a factor of 2, the luminosity will drop by a factor of $2^4 = 16$. And since the rate at which the white dwarf loses thermal energy equals L, the cooling proceeds ever more slowly. The coolest white dwarfs have $T_{eff} \sim 4000$ K, and have been cooling for $\sim 8 \times 10^9$ yrs. From the mass-radius graph we see that white dwarfs of mass 0.4, 0.8 and 1.25 $M_\odot$ will have radii of 0.015, 0.010 and 0.005 $R_\odot$, respectively. (0.01 $R_\odot$ is about the radius of the earth.) Since the luminosity varies as R^2 , if these stars had the same surface temperature, their luminosities would be in the ratios 2.25 : 1 : 0.25, with the most massive the faintest. This means that in an H-R diagram, the cooling tracks would be parallel with the more massive white dwarfs below the less massive. Furthermore, since the more massive stars have, at a given temperature, more internal energy but lower luminosity, they will evolve more slowly. The figure on the next page shows the cooling tracks in a "theorist's H-R diagram (i.e., $\log_{10} L$ vs $\log_{10} T_{eff}$).

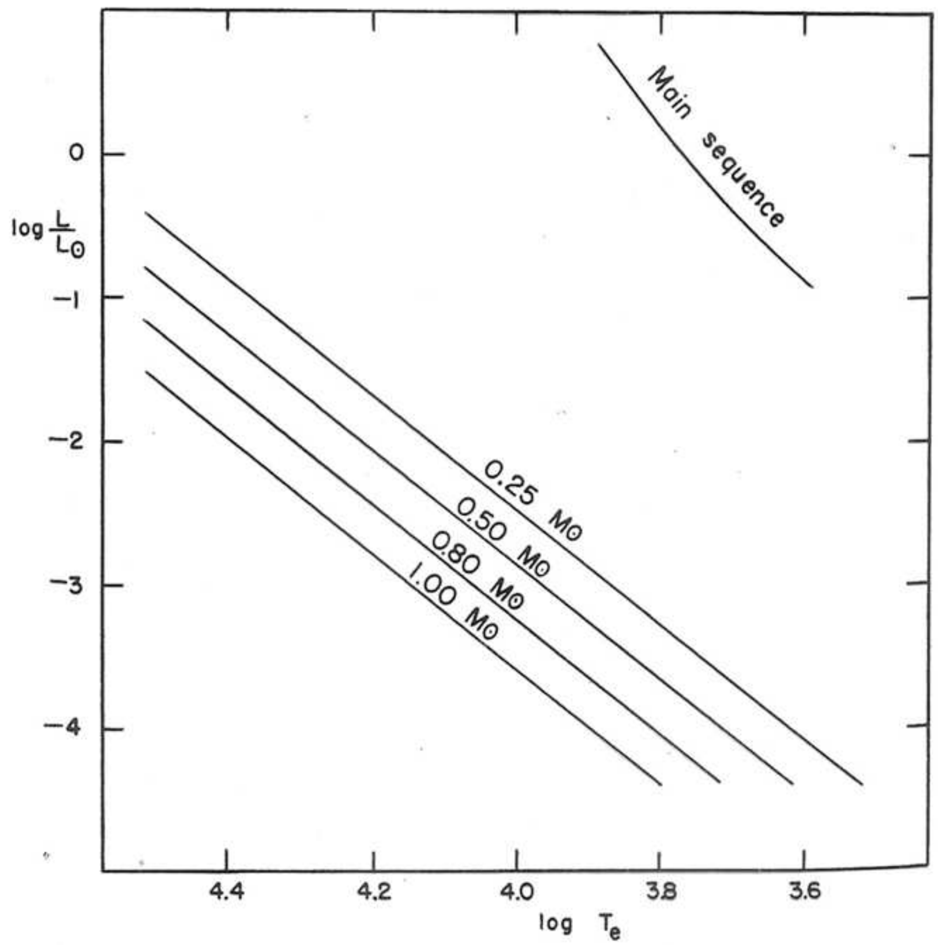


Fig. 26.1. Theoretical Hertzsprung-Russell diagram for the white dwarfs.

1.4. Neutron Stars

If a star should approach the Chandrasekhar mass limit, $M_{Ch} = 1.46\ M_\odot$, its radius will shrink and the density will increase as R^{-3} . At some point, new physics will intervene. A neutron in free space will live for only about 15 minutes before undergoing decay to a proton, electron and anti-neutrino: $n \rightarrow p\ +\ e^-\ +\ \overline{\nu_e}$. The reverse reaction $p\ +\ e^-\ \rightarrow n\ +\ \nu_e$ is also possible, but, since the neutron is more massive than the combined masses of the proton and electron, energy must be supplied for this reaction to occur. But if we consider a mixture of protons and electrons compressed to ever higher densities, the Fermi energy $\mathcal{E}_F$ of the electrons will at some point be high enough for the $p\ +\ e^-\ \rightarrow n\ +\ \nu_e$ reaction to proceed. At the same time, the neutron decay reaction will be blocked, because the quantum states into which the electron would go are all filled. This process of converting protons to neutrons would begin at $\rho \gtrsim 10^7$ g cm^{-3}. But the protons will not be free, but will be inside nuclei. When this is taken into account, the process of conversion to neutrons (called "neutron drip") is thought to take place for $\rho \gtrsim 3 \times 10^{11}$ g cm^{-3}.

We are thus led to contemplate a shrinking star where the electrons have been gobbled up by protons and the nuclei have dissolved into a dense neutron gas. What about the stability of such an object? Our first idea is to use the same equations as for the electrons: A degenerate neutron gas. We just replace the mass m in the white dwarf equations by the mass of the neutron. We also need to reconsider μ_e : μ_n will now be the mass per neutron, and thus 1 rather than 2. Looking at equation (14), we see that the mass limit for our ball of neutrons would be 5.8 $M_\odot$. That seems good – we can halt the collapse of our too-massive white dwarf. But we soon realize that we have not included important physics. For, if we make such a model with a mass of, say, 3.2 $M_\odot$, our white dwarf equations give us a value for the radius of $R = 7.7$ km, and a mean density of $< \rho >= 3.4 \times 10^{15}$ g cm^{-3}. But the radius of a 3.2 $M_\odot$ *black hole* is 9.5 km! Oops!

Clearly, we must use equations that take general relativity into consideration. This was done by Tolman and by Oppenheimer and Volkoff in 1939. The so-called *TOV equation* is

$$\frac{dP}{dr} \ = \ -\frac{G\ M_r}{r^2}\ \rho\ \left(1 + \frac{P}{\rho c^2}\right)\left(1 + \frac{3P}{< \rho > c^2}\right)\left(\frac{r}{r - r_s}\right) \qquad \text{where} \qquad (16)$$

$$\frac{dM_r}{dr} \ = \ 4\pi r^2\ \rho \qquad < \rho > \ = \ \frac{M_r}{\frac{4}{3}\pi r^3} \qquad r_s \ = \ \frac{2G\ M_r}{c^2} \qquad (17)$$

Here, r_s is just the Schwarzschild radius of a mass equal to that enclosed by radius r. If we neglect the last three terms of equation (16), you see it is just the equation of hydrostatic equilibrium. The first two of those last three factors accounts for the added mass due to energy in the form of pressure, while the last is a space-curvature correction.

So what happens when we use the TOV equation along with the full $P = A\ f(x)$ pressure equation for degenerate neutrons? Well, good news & bad news. We do get stable neutron stars, but the limiting mass is too small: $M_{max} = 0.7\ M_\odot$. Our models near this limit will have a radius of $R = 9.6$ km and a central density of $\rho_c = 5 \times 10^{15}$ g cm^{-3}.

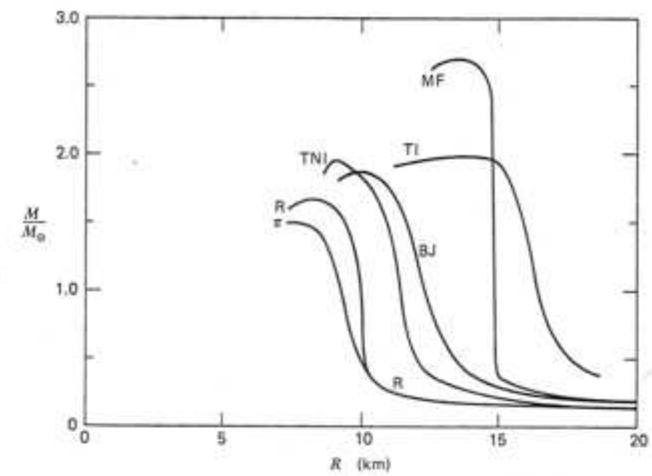
Since most *observed* neutron stars have masses around 1.4 $M_\odot$ – twice this "maximum" – something must be wrong. The problem is not hard to find. For our white dwarf models, we assumed the electrons were an ideal (i.e., non-interacting) gas; that was justified. But at densities of $\rho \sim 10^{15}$ g cm^{-3}, neutrons are *not* an ideal gas. These are the densities we find within an atomic nucleus, and the neutrons interact with one another via the strong force.

Thus we see that to model neutron stars we need the TOV equation and an equation of state that includes not only degeneracy but the nuclear forces between the neutrons. Unfortunately, we don't have any such definitive equation of state. Many approximate models have been made, and they do raise M_{max} quite a bit, so that we can model observed neutron stars. The next page shows some early models. Some of these models have already been shown to be inadequate by observations: We know of a neutron star (a pulsar in a close orbit to a white dwarf) with a well determined mass of 2 $M_\odot$. More exotic ideas abound, such as *quark stars* where, at sufficiently high densities, the neutrons dissolve into their constituent up and down quarks, and some become strange quarks ...

With such uncertainty, are we sure there is *any* mass limit to neutron stars? Yes, because there is a limit to the pressure that material can exert, regardless of the (unknown) equation of state. And it's really quite simple. The speed of sound, v_s, is related to the pressure and density by $v_s^2 = P/\rho$. But there is a limit to the sound speed: It must not exceed the speed of light, otherwise causality breaks down. So we must have $v_s < c$, regardless of the equation of state. This implies that P must be less than $\rho\, c^2$. So if we just set P to $\rho\, c^2$ in the TOV equation we have:

$$\frac{d\rho}{dr} \;=\; -\frac{G\, M_r}{c^2\, r^2}\, \rho\;\; (2) \left(1 + \frac{3\rho}{<\rho>}\right) \left(\frac{r}{r - r_s}\right) \quad . \tag{18}$$

If we make models by integrating this equation, we find $M_{max} \sim 5\ M_\odot$. And this is a real upper limit – things that exceed it we identify as black holes! Most theorists think the actual mass limit is less, $\lesssim 3\ M_\odot$. It may be that we turn the problem around, and are able to use *observed* masses and radii of neutron stars to pin down the nuclear equation of state.



The Maximum Mass of a Neutron Star For Various Equations of State

Equation of State[a]	Maximum Mass ($M_\odot$)
π	1.5
R	1.6
BJ	1.9
TNI	2.0
TI	2.0
MF	2.7

Fig. 2.— Some (old) neutron star models with different equations of state. Note that models leftward of the peak in the mass-radius curve are unstable – the peak is the model with the maximum mass.