Photoionized Gas: Ionization Equilibrium; the Strömgren Sphere.

1 Ionization and Recombination.

Two circumstances lie behind most of the physical processes peculiar to $H^+$ regions\(^1\) and planetary nebulae: the low density of the gas and extreme dilution of the radiation. As we shall see, this leads to a situation where the ionization balance and the population of the atomic levels is far from thermodynamic equilibrium.

The densities in $H^+$ regions are typically $10^{-10}$ H particles cm\(^{-3}\), while densities for planetaries are somewhat higher, say $10^2 - 10^4$ cm\(^{-3}\).

The intensity of the stellar radiation at a distance $D$ from the star is described in terms of the dilution factor, $W$, which is the fraction of the sphere which the stellar surface subtends as observed from distance $D$:

\[
W = \frac{1}{2} \left[ 1 - \sqrt{1 - \left( \frac{R_*}{D} \right)^2} \right] \approx \frac{1}{4} \left( \frac{R_*}{D} \right)^2 ,
\]

where $R_*$ is the stellar radius. Typical values for a planetary nebula might be

\[
R_* \sim 0.3 \, R_\odot \sim 2 \times 10^{10} \text{ cm} ; \quad D \sim 0.1 \text{ pc} \sim 3 \times 10^{17} \text{ cm} ,
\]

so that

\[
W \sim 10^{-15} .
\]

Thus we see that the dilution of the radiation is quite extreme.

Consider the behavior of hydrogen in such a radiation field. The hot star is rich in radiation with wavelengths shorter than 912 Å, which is able to ionize the gas; conversely, the electrons and protons continually recombine:

\[
H^0 + h\nu \rightarrow H^+ + e^- \rightarrow H^0 + h\nu
\]

Thus the fundamental processes we must consider are photoionization and recombination. Now in the case of discrete atomic levels, we may describe radiative excitation and de-excitation by the Einstein coefficients $A_{ul}$, $B_{ul}$, and $B_{tu}$. These coefficients are not independent, but obey the familiar relations

\[
A_{ul} = \frac{2h\nu^3}{c^2} B_{ul} = \frac{2h\nu^3}{c^2} \frac{g_l}{g_u} B_{tu} ,
\]

where the subscripts $u$ and $l$ refer to the upper and lower levels, and $g_u$ and $g_l$ are the respective statistical weights. These relations may be derived by considering the atom to be immersed in an equilibrium radiation field given by the Planck function:

\[
B_{\nu} = \frac{2h\nu^3}{c^2} \frac{1}{e^{h\nu/kT} - 1} .
\]

\(^1\)I prefer “$H^+$ region” to the more common “H II region”, since the latter is, strictly speaking, a spectroscopic notation and $H^+$ has no spectrum. It also sounds the same as “$H_2$ region”, i.e., a region of molecular hydrogen.
When the atom, or rather the collection of atoms, reaches a steady state, the population of the levels must be related by the Boltzmann equation:

\[
\frac{N_u}{N_l} = \frac{g_u}{g_l} e^{-\frac{E_u - E_l}{kT}},
\]

(4)

Here, \( E_u \) and \( E_l \) are the energies of the upper and of the lower states. Exactly analogous relations must exist between the coefficients of photoionization and recombination, although the notation used for the continuum states looks rather different.

The upward transitions may be described by the continuous absorption coefficient, \( a(\nu) \), which can be defined as follows:

\[
dE = 4\pi I_\nu \ N_{r,n} \ a_{r,n}(\nu) \ d\nu,
\]

(5)

where \( dE \) is the energy absorbed (ergs s\(^{-1}\) cm\(^{-3}\) in the frequency interval \( \nu \) to \( \nu + d\nu \), \( r \) refers to the stage of ionization, \( n \) to the specific atomic level, and the intensity \( I_\nu \) is measured per steradian, which accounts for the factor of \( 4\pi \).

The downward transitions, corresponding to the Einstein A coefficient, are described by a cross-section for recombination, \( \sigma_{r,n}(\nu) \), which is a function of the electron velocity \( \nu \), i.e., the continuum energy level occupied by the electron. The number of recaptures cm\(^{-3}\) s\(^{-1}\) will be

\[
N_e \ N_{r+1} \ \nu \ \sigma_{r,n}(\nu) \ f(\nu) \ d\nu,
\]

(6)

for electrons in the velocity range \( \nu \) to \( \nu + d\nu \), which result in an r-times ionized atom in level \( n \). Here, \( f(\nu) \) is the fraction of electrons with velocity \( \nu \).

Finally, the presence of radiation \( I_\nu \) will stimulate recombinations at a rate

\[
I_\nu \ N_e \ N_{r+1} \ \nu \ \sigma'_{r,n}(\nu) \ f(\nu) \ d\nu,
\]

(7)

where we can call \( \sigma'_{r,n} \) the cross-section for stimulated recombination.

If we immerse the atoms in a blackbody radiation field, the rate of ionizations from level \( n \) to velocity \( \nu \) must equal the inverse rate - the principle of detailed balance:

\[
\frac{4\pi B_\nu}{h\nu} \ N_{r,n} \ a_{r,n}(\nu) \ d\nu = N_e \ N_{r+1} \ \nu \ f(\nu) \left[ \sigma_{r,n}(\nu) + B_\nu \ \sigma'_{r,n}(\nu) \right] \ d\nu.
\]

(8)

The electrons will have a Maxwellian distribution of velocities characterized some temperature \( T \) (we will justify this equilibrium distribution later),

\[
f(\nu) = 4\pi (m/2\pi kT)^{3/2} \ \nu^2 \ e^{-\frac{m\nu^2}{2kT}}.
\]

(9)

If we insert this into (8) and note that

\[
h\nu = \frac{1}{2} \ m \ \nu^2 + (I_r - E_{r,n}),
\]

(10)

where \( I_r \) is the ionization potential of the r-times ionized atom, it is possible to reduce the expression to the following form:

\[
\frac{N_e \ N_{r+1}}{N_{r,n}} = \frac{(2\pi m kT)^{3/2}}{h^3} \ a_{r,n}(\nu) \ \frac{2h^2 \nu^2}{\sigma_{r,n}(\nu) \ c^2 \ m^2 \nu^2} \ e^{\frac{E_{r,n}-I_r}{kT}} \ \left\{ 1 - e^{-\frac{h\nu}{kT}} \left[ 1 - \frac{2h\nu^3 \ \sigma'_{r,n}(\nu)}{c^2 \ \sigma_{r,n}(\nu)} \right] \right\}^{-1}
\]

(11)
But the equilibrium state must in fact be that given by the combined Boltzmann and Saha equations,
\[
\frac{N_e}{N_{r,n}} = \frac{(2\pi m k T)^{3/2}}{h^3} \frac{2 U_{r+1}(T)}{g_{r,n}} e^{\frac{E_{r,n} - \epsilon}{k T}} ,
\]
regardless of the temperature $T$. Here, $U_{r+1}(T)$ is the partition function of the recombining ion, and will depend upon the relative numbers of these ions in various states of excitation, which in turn depends upon the temperature. Usually, however, it will be very well approximated by the statistical weight of the ground state. Now the quantities $a_{r,n}(\nu)$ and $\sigma_{r,n}(\nu)$ are atomic parameters and can know nothing about a macroscopic quantity like temperature, but only know frequency and velocity. Thus (11) can be equal to (12) for all values of $T$ only if
\[
\frac{a_{r,n}(\nu)}{\sigma_{r,n}(\nu)} = \frac{c^2 m^2 \nu^2}{h^2 \nu^2} \frac{U_{r+1}(T)}{g_{r,n}}
\]
and
\[
\frac{\sigma'_{r,n}(\nu)}{\sigma_{r,n}(\nu)} = \frac{c^2}{2 h \nu^3} .
\]

The first expression is called Milne’s relation. The second is just equivalent to the relation between the Einstein $A_{\nu,\ell}$ and $B_{\nu,\ell}$. (Stimulated recombination is not usually discussed, not because it does not exist, but because under nebular conditions of highly dilute radiation it is usually of no importance.)

Now, if we want the total number of recombinations cm$^{-3}$ s$^{-1}$ to some level $n$ of the $r$ stage of ionization, we must integrate the cross section over the full range of velocities of the free electrons:
\[
N_e N_{r+1} \alpha_{r,n}(T) = \int_0^\infty N_e N_{r+1} \nu \sigma_{r,n}(\nu) f(\nu) d\nu ,
\]
where $\alpha_{r,n}(T)$ is called the recombination coefficient. In most situations, the electron velocity distribution function $f(\nu)$ is Maxwellian. Then, substitution of (9) and (13) into (15) results in an expression which may be used to compute the recombination coefficients from the (in some cases) known photoionization cross-sections:
\[
\alpha_{r,n}(T) = \frac{1}{(2/\pi)^{1/2}} e^{\frac{-E_{r,n}}{k T}} \frac{g_{r,n}}{U_{r+1}(T)} \int_{(I_r - E_{r,n})}^\infty (\nu \nu) e^{-\frac{\nu^2}{k T}} a_{r,n}(\nu) d(\nu) .
\]

The photoionization cross-sections $a_{r,n}(\nu)$ are only known exactly for the case of one-electron systems like $H^0$, $He^+$, etc., and even then the expressions are quite complex. For other atoms and ions, we have the results of numerical calculations or laboratory measurements. It follows that to compute the recombination coefficients from equation (1.17) we must evaluate the integral numerically. However, we can do the integral easily if we assume that the photoionization cross-section has the special form $a_{r,n}(\nu) = a_T (\nu_0 \nu)^2$, where $\nu_0$ is the threshold frequency where photoionization begins, defined by $\nu_0 = (I_r - E_{r,n})$, and $a_T$ is the cross-section at threshold. In this case we find that equation (16) leads to
\[
\alpha_{r,n}(T) = \frac{1}{(2/\pi)^{1/2}} e^{\frac{-E_{r,n}}{k T}} \frac{g_{r,n}}{U_{r+1}(T)} (I_r - E_{r,n})^2 a_T .
\]
2 Interstellar conditions compared to thermodynamic equilibrium.

Now let us consider a star which radiates as a blackbody at some temperature $T_\ast$. Then the flux from each cm$^{-2}$ of the stellar surface is

$$\pi B_\nu = \frac{2\pi \hbar \nu^3}{c^2} \frac{1}{e^{\hbar \nu/kT_\ast} - 1}. \tag{18}$$

The mean intensity per steradian, $J_\nu$, in the nebula is

$$J_\nu = \frac{1}{4\pi} \int I_\nu \, d\Omega = W \, B_\nu, \tag{19}$$

where the dilution factor $W$ is given by (1). If we consider the radiative excitations and de-excitations in an atom exposed to this radiation field we can use the Einstein A and B coefficients to write

$$R = \frac{\text{(rate upward)}}{\text{(rate downward)}} = \frac{J_\nu \, B_{lu}}{A_{ud} + J_\nu \, B_{ul}}. \tag{20}$$

Using (2) and (18), we see this can be written as

$$R = \frac{W \, (g_u/g_l)}{\exp(h\nu/kT_\ast) - 1 + W} \approx \frac{W \, (g_u/g_l)}{\exp(h\nu/kT_\ast) - 1}. \tag{21}$$

Thus the downward transitions proceed over $W^{-1}$ times faster than the radiative excitations, and we expect to find virtually all atoms in their ground states. (Exceptions to this are metastable levels, where transitions to the ground state are forbidden by selection rules. If such a level is populated by transitions from other, higher levels, a significant population may build up. The The $2^{3}S$ state of helium is the classic example.)

As a consequence of the foregoing, when we consider the balance of ionizations and recombinations in the gas, we need only consider ionizations from the ground level. Recombinations occur to all levels, however. We can now write the fundamental equation which determines the degree of ionization of hydrogen:

$$N_e \, N(H^+) \sum_{n=1}^{\infty} \alpha_n(T) = N(H^0) \int_{\nu_0}^{\infty} \frac{4\pi \, J_\nu}{h\nu} \, a_1(\nu) \, d\nu, \tag{22}$$

where $a_1(\nu)$ is the continuous absorption (photoionization) coefficient from the ground state and $\nu_0$ is the threshold for ionization from this level, $h\nu_0 = 13.595$ eV. This equation is quite general.

Let us now consider the special case of dilute blackbody radiation. Then with (18) and (19), (22) becomes

$$\frac{N_e \, N(H^+)}{N(H^0)} = W \, \frac{\int_{\nu_0}^{\infty} \frac{8\pi \nu^2}{c^2} \frac{a_1(\nu) \, d\nu}{\exp(h\nu/kT_\ast) - 1}}{\sum_{n=1}^{\infty} \alpha_n(T)}. \tag{23}$$
Normally, the electron temperature of the gas, $T$, will be considerably less than the stellar temperature, $T_*$. Let us compare equation (23) to the Saha equation evaluated at the stellar temperature:

$$\left[ \frac{N_e \ N(H^+)}{N(H^0)} \right]_{\text{Saha}} = \frac{(2\pi m k T_*)^{3/2}}{\hbar^3} e^{-I/k T_*}.$$  \hfill (24)

(Note that we have dropped the subscript $r$ referring to the stage of ionization; $I$ is the ionization potential of hydrogen.) If we insert (16) and (24) into (23), we can write the result in the form

$$\frac{N_e \ N(H^+)}{N(H^0)} = W \ Q \ \left[ \frac{N_e \ N(H^+)}{N(H^0)} \right]_{\text{Saha}},$$ \hfill (25)

where $Q$ is given by the messy expression

$$Q = \left( \frac{T}{T_*} \right)^{3/2} \ \frac{e^{I/k T_*} \ \int_0^\infty (h \nu)^2 \ \frac{a_1(\nu)}{\exp(h \nu/k T) - 1} d(\nu)}{\sum_{n=1}^\infty \ \frac{g_n}{T} \ e^{(I - E_n)/k T} \ \int_0^\infty (h \nu)^2 \ a_n(\nu) \ e^{-h \nu/k T} d(\nu)}. \hfill (26)$$

Letting $x = h \nu/k T_*$, $x_0 = I/k T_*$, $z = h \nu/k T$, and $z_n = (I - E_n)/k T$, this becomes

$$Q = \left( \frac{T}{T_*} \right)^{1/2} \ \frac{e^{x_0} \ \int_0^\infty x^2 \ \frac{a_1(\nu)}{e^x - 1} dx}{\sum_{n=1}^\infty \ \frac{g_n}{2} \ e^{z_n} \ \int_0^\infty e^{-z} \ z^2 \ a_n(\nu) \ dz}. \hfill (27)$$

A large part of the sum comes from the first term, recombinations to the ground state. The whole expression is of order unity. The continuous absorption coefficient of hydrogen is approximately $a_n(\nu) = a_0 \ n^{-5} \ \nu^{-3}$. If instead, we assume a frequency dependence of $a_n(\nu) = a_0 \ n^{-5} \ \nu^{-2}$, we can evaluate the integrals easily. For hydrogen the statistical weights are $g_n = 2 \ n^2$. We then find that we can evaluate the infinite series and we finally obtain:

$$Q = \left( \frac{T}{T_*} \right)^{1/2} \ \frac{e^{x_0}}{1.202} \ \ln \left[ \frac{1}{1 - e^{-x_0}} \right]. \hfill (28)$$

Take, for example, $T = 10^4$ and $T_* = 10^5 \ K$. Then $x_0 = 1.58$, which yields $Q = 0.6$. The point we are trying to establish is that, analogously with the case of the bound levels,

$$\frac{N_e \ N(H^+)}{N(H^0)} \approx W \ \left[ \frac{N_e \ N(H^+)}{N(H^0)} \right]_{\text{Saha}}. \hfill (29)$$

We see that in gaseous nebulae, the ground state is vastly overpopulated with respect to both the excited levels and with respect to the continuum states of ions plus electrons, when compared to thermodynamic equilibrium conditions. This does not mean that the degree of ionization is small, however. Because of the extremely low densities (as compared with a stellar atmosphere, for example) the material in a gaseous nebula can be 99.9% ionized and still be $10^{15}$ times less ionized than would be predicted by the Saha equation.
3 Relaxation Time of Electrons due to Coulomb Encounters.

We mentioned above that we would justify the use of the Maxwellian velocity distribution. The electron velocities reach equilibrium because the elastic interactions between them are relatively fast when compared to the other processes which could act to destroy the equilibrium distribution. Let us first consider the Coulomb encounters between two electrons. In the center-of-mass frame, an electron approaches the scattering center with impact parameter $s$ and is scattered through an angle $\theta$. The relationship is

$$ s = \frac{e^2}{2E} \cot \left( \frac{\theta}{2} \right), \quad (30) $$

where $E$ is the kinetic energy of the encounter and $e$ is the electron charge (see e.g., Goldstein, Classical Mechanics, page 83).

We will find that most of the energy is lost in the many deflections through small angles, rather than in the few deflections through large angles. Thus we can take $\theta \ll 1$. Then, since $\cot(x) = \frac{1}{x} - \frac{x}{3} + \cdots$, we have

$$ s \approx \frac{e^2}{E \theta}, \quad |ds| = \frac{e^2 d\theta}{E \theta^2}. \quad (31) $$

Let $dN$ be the number of electrons scattered into the range $\theta$ to $(\theta + d\theta)$ per unit time. Then

$$ dN = N_e \nu \left( 2\pi s ds \right) = N_e \frac{2^{3/2} \pi e^4}{E^{3/2} m^{1/2} \theta^3}, \quad (32) $$

where we have used $\nu = (2E/m)^{1/2}$. The deflection through angle $\theta$ represents a change in the kinetic energy $E$, which we will call $\Delta E$. Then since $v_\perp = v \sin \theta \approx v \theta$, $\Delta E = \frac{1}{2}mv_\perp^2 \approx \theta^2 E$.

(This is a swindle, and only works when the two particles are of equal mass, but I don’t want to go through the coordinate change from center-of-mass to laboratory.) So now we can write down the rate of energy loss per unit time due to Coulomb encounters:

$$ \frac{dE}{dt} = - \int_{\theta_{\text{min}}}^{\theta_{\text{max}}} \Delta E \ dN = - N_e \frac{2^{3/2} \pi e^4}{E^{3/2} m^{1/2} \theta^3} \frac{1}{\sqrt{mE}} \ln \left[ \frac{\theta_{\text{max}}}{\theta_{\text{min}}} \right]. \quad (33) $$

We can then estimate the time required for the suprathermal electron of energy $E$ to lose its excess energy and slow to thermal velocities with energy $E_{th}$:

$$ t_{\text{step}} = \int_{E}^{E_{th}} \left( \frac{dE}{dt} \right)^{-1} dE = \frac{m^{1/2} \left[ E^{3/2} - E_{th}^{3/2} \right]}{3 \sqrt{2} \pi e^4 N_e \ln \Lambda}, \quad (34) $$

where we have defined $\Lambda = (\theta_{\text{max}}/\theta_{\text{min}})$. We can just let $\theta_{\text{max}} = \pi$. Now $\theta_{\text{min}}$ corresponds to $s_{\text{max}}$, the largest impact parameter that is to be considered. We can’t just let $s \to \infty$ because the
result is unbounded. This is not reasonable in any case, because of the effects of other particles. We could just cut off \( s \) at the value that corresponds to the mean particle separation, \( d = N_e^{1/3} \). For example, at \( N_e = 100, d = 0.2 \text{ cm} \). A better choice is the Debye shielding length (e.g. Shu, ‘Gas Dynamics’, p8),

\[
L_D = \sqrt{\frac{kT}{4\pi \epsilon^2 (N_e + Z^2_i n_i)}} = 4.88 \sqrt{T/N_e},
\]  
(35)

for pure hydrogen, where \( Z_i = 1 \) and \( n_i = N_e \). For example, at \( T = 10^4 \) and \( N_e = 100, L_D = 49 \text{ cm} \). Because we are evaluating \( \ln(\Lambda) \), the choice doesn’t matter much. Using the Debye length and (31), we find that

\[
\Lambda = 1.06 \times 10^8 E \sqrt{T/N_e},
\]  
(36)

where \( E \) is measured in eV. Note that \( \Lambda \) depends on \( E \) — when we integrated to obtain (34), we neglected the logarithmic energy dependence of \( \epsilon \).

As an example, consider a nebula with \( T = 10^4, N_e = 100, \text{ and } E = 10 \text{ eV} \). Then \( \Lambda = 1.06 \times 10^{10} \) and \( \ln(\Lambda) = 23.1 \). This corresponds to scattering through an angle of only \( \theta_{\text{min}} = 2 \times 10^{-10} \) radians!

Evaluating the constants in equation (34), and assuming that \( E_{\text{th}} \ll E \), we obtain

\[
t_{\text{stop}} \approx \frac{86300 E^{3/2}}{N_e \ln(\Lambda)} \text{ sec},
\]  
(37)

where \( E \) is in eV. For the example given above \( (T = 10^4, N_e = 100, E = 10 \text{ eV}) \), we find that \( t_{\text{stop}} = 1200 \text{ sec} = 20 \text{ min} \).

This may seem a long while, but compare this to the time needed for a recombination to occur: this will be given by

\[
t_{\text{rec}} = \frac{1}{N_e \alpha_A},
\]  
(38)

where \( \alpha_A \) is the total recombination coefficient for hydrogen, which at 10000 K has a numerical value of \( \alpha_A \approx 4 \times 10^{-13} \text{ sec} \). Evaluating this for the same conditions, we find that \( t_{\text{rec}} = 2.5 \times 10^{10} \text{ sec} = 830 \text{ yr} \). Thus when an electron is ejected by photoionization with a suprathermal energy, it shares this energy with the other electrons long before it has a chance of recombining. So the particle distribution is very nearly Maxwellian.

But note that the stopping time increases strongly with energy, so if the particle is sufficiently energetic, and if in addition the gas is mostly neutral so that the electron density is low, thermalization may be so slow that an important suprathermal electron population may exist.
4 Distribution of Ionized Gas Around a Star.

As stellar radiation penetrates into the $H^+$ region around a star, it is not only diluted by geometry, but is weakened by absorption. This absorption may finally limit the size of the ionized region. The nature of the transition from ionized to neutral gas was investigated by Strömgren (1939) by numerical integration. The basic result can be seen in a simpler treatment by Seaton (1960) which uses the plane-parallel approximation. By neglecting the increasing dilution as we move away from the star, we only underestimate the sharpness of the transition, and not by much. We let $F$ be the flux in number of ionizing photons cm$^{-2}$ sec$^{-1}$ which flow in the direction $x$:

$$F = \int_{\nu_0}^{\infty} \frac{I_{\nu}}{h\nu} \, d\nu .$$

(39)

A more serious approximation must be made: we neglect the frequency dependence of the absorption coefficient and instead introduce a mean continuous absorption coefficient $\bar{a}$. The change in $F$ due to the absorption of photons by neutral H atoms is thus

$$\frac{dF}{dx} = - \bar{a} \, N(H^0) \, F .$$

(40)

The equation of ionization equilibrium is given by

$$N(H^0) \, F \, \bar{a} = \alpha_A \, N(H^+) \, F^2 ,$$

(41)

where $\alpha_A = \sum_{\alpha=1}^{\infty} \alpha_a$ and we have made use of the fact that $N_e = N(H^+) \, y$, since the electrons in the nebula (assumed pure hydrogen) come from the ionization of hydrogen. We now define the total hydrogen density $\bar{N} = N(H^0) + N(H^+)$ and $y = N(H^0)/N$. Then (40) and (41) become

$$\frac{dF}{dx} = - \bar{a} \, N \, y \, F$$

(42)

and

$$F = \frac{\alpha_A \bar{N}}{\bar{a}} \, \frac{(1 - y)^2}{y} .$$

(43)

Inserting (43) into (42) gives the equation

$$\frac{1}{\bar{N} \bar{a}} \frac{d}{dx} \left[ \frac{(1 - y)^2}{y} \right] = \frac{1}{y} \left( 1 - \frac{1}{y} \right) .$$

(44)

We see that the recombination coefficient has dropped out. We next replace the distance $x$ with the variable $\tau = \bar{N} \bar{a} x$. (This would be the optical depth if the gas were completely neutral.) Then (44) reduces to

$$\frac{dy}{d\tau} = y \frac{1 - y}{1 + y} ,$$

(45)

which can be easily integrated to obtain

$$\tau(y) = 2 \ln y - 2 \ln(1 - y) - \frac{1}{y} + C ,$$

(46)

where $C$ is a constant which depends upon the point from which we choose to measure $\tau$ (or $x$). Consider the point where $y = 1/2$, i.e., $N(H^+) = N(H^0)$. Then we see that $\tau(y = 0.5) = C - 2$. 

8
Thus if we measure $\tau$ from this point, that is, if $\tau(0.5) = 0$, we must set $C = 2$. With this choice of origin, we can write the solution as

$$\tau(y) = 2 + 2 \ln \left[ \frac{y}{1-y} \right] - \frac{1}{y}. \quad (47)$$

The following table gives some numerical values of this function, and there is a plot of it on the next page.

<table>
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<th>$y = N(H^0)/N$</th>
<th>$\tau = Na x$</th>
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<td>0.005</td>
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</tr>
<tr>
<td>0.010</td>
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</table>

The interval from 95% ionized to 5% ionized occurs over an interval of $\Delta \tau \simeq 30$. The corresponding physical distance is therefore

$$\Delta x \simeq \frac{30}{Na} \text{ cm}. \quad (48)$$

If we take $\bar{a} = 6 \times 10^{-18}$ cm$^2$, the value near the absorption threshold, and express the distance in parsecs, we obtain

$$\Delta x \simeq \frac{2}{N} \text{ parsecs}, \quad (49)$$

where the hydrogen density, $N$, is the number cm$^{-3}$. Consider an $H^+$ region with $N = 100$. Equation (49) then says that the thickness of the 95% → 5% transition zone is only 0.02 pc, much smaller than the radius of the ionized region. This was the result of Strömgren: a hot star should be surrounded by a region which is almost completely ionized, and bounded by a very sharp transition zone. Such a structure is called a Strömgren sphere.

It is now recognized that the interstellar medium is ionized to some extent in regions other than the bright $H^+$ regions. Some of this partial ionization may be due to X-rays. If we consider ionization by 200 eV X-rays, then the absorption coefficient at such energies is smaller by a factor of about $(13.6/200)^3 = 3 \times 10^{-4}$, so that the thickness of the transition region would become $\Delta x \approx 5000/N$ pc. Thus there would not be a sharp boundary to such a region of ionization, but rather a slow decline of the ionization fraction.

Photographs of $H^+$ regions and of planetary nebulae often show sharp edges. There are two possibilities: either the edge is the sharp transition zone from $H^+$ to $H^0$, or the edge represents a sharp drop in density, an actual edge of the cloud. In the first case, the region is called ionization bounded, while the second case is termed density bounded. In practice, it is often quite difficult to be sure which type of edge we are observing.
5 Absorption in the Hydrogen Lines.

In our treatment of the edge of the ionized region, we have assumed that all the ionizing radiation comes from the star. But when an electron recombines directly to the ground state, a Lyman continuum photon is released with $\lambda \leq 912 \text{Å}$, which can ionize another hydrogen atom. So let us consider the fate of the line and continuum photons within the gas.

We have seen that our nebula will consist of a well defined region of highly ionized gas, with a small number of neutral atoms sufficient to satisfy the ionization equation (22). Typically, this will be $N(H^0)/N(H^+) = 10^{-3} - 10^{-4}$. The constant recombination to excited levels, followed by downward transitions ("cascades") gives rise to the hydrogen emission lines so prominent in the spectra of all gaseous nebulae. We can observe the whole Balmer series in the visible part of the spectrum; the Paschen lines in the infrared are also routinely observed. Recombinations directly to the ground state (which comprise something like 40% of the total recombinations) give rise to an ionizing continuum. These continuum photons cannot escape any more easily than the stellar ionizing radiation; in fact, they will escape less easily, since they will be concentrated near the photoionization threshold (most will be within 1 eV of it if the gas is near $10^4 \text{ K}$) where the absorption by hydrogen is the strongest.

Let us consider the line radiation. Since nearly all the atoms are in the ground state, lines produced by jumps that terminate on any level except $n=1$ will see no line opacity and thus will escape from the nebula without absorption (unless absorbed by some other agent such as dust). But transitions to the level $n=1$ (the Lyman lines) will generally see considerable opacity and are likely to be reabsorbed by the gas. Let us look at this quantitatively. The absorption coefficient per atom for any atomic line can be written as

$$a_{lu}(\nu) = \frac{\pi e^2}{mc} f_{lu} \phi(\nu) = 0.02654 f_{lu} \phi(\nu),$$

where $\phi(\nu)$ is the line profile function, normalized to unity and $e$ and $m$ are the electron charge and mass, respectively. The quantity $f_{lu}$, called the f-value, is related to the Einstein A value of the transition by

$$A_{ul} = \frac{g_l}{g_u} 6.670 \times 10^{15} \lambda^{-2} f_{lu}, \quad \text{with } \lambda \text{ in Å}. \quad (51)$$

The $g$’s are statistical weights. The f-values of strong lines are of the order of unity (e.g., for L$\alpha$, $f = 0.416$). In the case of the lines in gaseous nebulae, there are not enough collisions to broaden the lines; instead the profile shape is determined by the Doppler shifts due to thermal motions of the atoms. Thus

$$\phi(\nu) = \frac{1}{\sqrt{\pi} \Delta \nu_D} \exp \left[-\left(\frac{\nu - \nu_0}{\Delta \nu_D}\right)^2\right], \quad (52)$$

where the **Doppler width** of the line is

$$\Delta \nu_D = \sqrt{\frac{2kT}{M}} \frac{1}{\lambda_0}, \quad (53)$$

and $M$ is the mass of the atom. $\lambda_0$ and $\nu_0$ are the central wavelength and frequency of the line. For H atoms at 10000 K, $\Delta \nu_D = 1.06 \times 10^{11} \text{ Hz}$, or $\Delta \lambda_D = 0.09 \text{ Å}$. Thus the amount of the star’s radiation which can be absorbed by the Lyman series is very small compared to the amount which is absorbed by the Lyman continuum. But since each photoionization is balanced by a recombination to some level $n$, the atoms arrive at $n$ almost exclusively by recombinations or by
transitions from higher levels, and not by direct excitation by absorption of stellar radiation in the Lyman lines. Hence we will neglect line absorption of stellar radiation.

The probability that a Lyman line photon emitted by the gas will be absorbed will clearly depend upon the optical depth of the nebula in that line. The line optical depth of the nebula can be found from (50) if we know the number of $H^0$ atoms. To sidestep the need to solve for the number of $H^0$ atoms, we will compare the absorption coefficient of the Lyman lines at line center with the continuous absorption coefficient at the absorption threshold, which is also proportional to $N(H^0)$:

<table>
<thead>
<tr>
<th>n (upper level)</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>...</th>
<th>10</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ratio(line/continuum)</td>
<td>9350</td>
<td>1500</td>
<td>520</td>
<td>240</td>
<td>135</td>
<td>...</td>
<td>27</td>
</tr>
</tbody>
</table>

The results depend a bit on the temperature through (53); these values are for $10^4$ K. The basic result is the following: If the gas cloud is dense or extensive enough that the continuum optical depth is of order unity - which will be the case unless the nebula extremely faint - then the optical depth in the Lyman lines is very high and the Lyman line photons have almost no chance of escaping without repeated absorptions.

Now let us consider the chain of events started by a single stellar ionizing photon. The photon is absorbed and ejects a photoelectron. In a state of statistical equilibrium, that ionization must be balanced by a recombination. The recombination can either be

a) to the ground state, emitting an ionizing photon, or

b) to an excited state, emitting a continuum photon that cannot ionize hydrogen.

If case (a) should occur, the photon produced will be absorbed, ejecting a photoelectron, which once again must be balanced by a recombination, leading back again to the choice between (a) and (b). The chain of events started by the stellar photon must thus eventually lead to (b); the only question is whether this happens immediately or after a series of absorptions.

Having arrived at (b) - recombination to an excited level - the electron makes one or more jumps downward, eventually reaching the ground level. Let us consider the last of these jumps, which can originate on any level $n$, but will terminate on $n = 1$ and thus produce a Lyman line photon.

We saw above that the nebula will be very opaque in the Lyman lines, so this line photon will be reabsorbed, exciting some other atom back to the same level $n$. Now, unless $n = 2$, there are two possibilities for this excited atom:

c) jump again to $n = 1$, re-emitting the Lyman line that was just absorbed, or

d) jump to some level $n' < n$, emitting a line photon which will escape the nebula.

If (c) should occur, the new Lyman line photon will also be absorbed, leading once again to the choice between (c) and (d). Eventually, (d) must occur. But the atom, now in state $n'$, faces the same situation as when we considered arrival at state $n$. So ultimately, the events must lead to an atom excited to $n = 2$, and such an atom can emit only a Lyman $\alpha$ photon ($n = 2 \rightarrow n = 1$). This photon will just be scattered about until it finally works its way out of the nebula or until it runs into a dust particle and is destroyed.

We have thus shown that every stellar ionizing photon

(i) results in a recombination to an excited level, and

(ii) produces one Lyman $\alpha$ photon.

But for (ii) to occur, the previous step must have been a jump from either the continuum or from some level $n > 2$ down to $n = 2$. But such a jump corresponds to the emission of either a Balmer continuum photon or a Balmer line photon. Thus there is a third result of every stellar ionizing photon emitted, which is
(iii) production of exactly one Balmer line or continuum photon.

The above result is quite useful, for it means that if we count up all the Balmer line plus Balmer continuum photons — which are in the visible part of the spectrum — we have also counted all the stellar ionizing photons — photons which cannot be directly observed. Zanstra showed that by comparing the far ultraviolet flux of the star (obtained in this way from the Balmer radiation of the nebula) with the star’s directly observed visible flux, we can estimate the temperatures of the very hot stars which ionize $H^+$ regions and planetary nebulae. Such a temperature determination is called a **Zanstra temperature**.