A Survey of Geometric Algebra and Geometric Calculus

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(The current version is always available at my web page.)

> The principal argument for the adoption of geometric algebra is that it provides a single, simple mathematical framework which eliminates the plethora of diverse mathematical descriptions and techniques it would otherwise be necessary to learn. [10]

I have published two books:
Linear and Geometric Algebra: http://faculty.luther.edu/~macdonal/laga/
Vector and Geometric Calculus: http://faculty.luther.edu/~macdonal/vagc/

1 Foundations

1.1 Introduction

Geometric algebra and its extension to geometric calculus unify, simplify, and generalize many areas of mathematics that involve geometric ideas. They also provide a unified mathematical language for physics, engineering, and the geometrical aspects of computer science (e.g., graphics, robotics, computer vision).

This paper is an introduction to geometric algebra and geometric calculus, presented in the simplest way I could manage, without worrying too much about completeness or rigor. The only prerequisite is an understanding of undergraduate mathematics. In a few inessential sections some knowledge of physics is helpful.

My purpose is to demonstrate some of the scope and power of geometric algebra and geometric calculus. I will illustrate this for linear algebra, multivariable calculus, real analysis, complex analysis, and several geometries: euclidean, noneuclidean, projective, and conformal. I will also outline several applications.

Geometric algebra is nothing less than a new approach to geometry. Geometric objects (points, lines, planes, circles, ...) are represented by members of an algebra, a *geometric algebra*, rather than by equations relating coordinates. Geometric operations on the objects (rotate, translate, intersect, project, construct the circle through three points, ...) are then represented by algebraic operations on the objects.

Geometric algebra is *coordinate-free*: coordinates are needed only when specific objects or operations are under consideration.

All this has significant advantages over traditional approaches such as synthetic and analytic geometry and vector, tensor, exterior, and spinor algebra and calculus. The advantages are similar to those of elementary algebra over arithmetic: elementary algebra manipulates symbolic representations of numbers independently of their values, and geometric algebra manipulates symbolic representations of geometric objects independently of their coordinates in some coordinate system.

Efficient algorithms have recently been developed for implementing geometric algebra on computers [6].

At first you will likely find the novelty and scope of the mathematics presented here overwhelming. This is to be expected: it takes years of serious study to understand the standard approaches to the mathematics discussed here. But after some study I hope that you find, with me, great unity, simplicity, and elegance in geometric algebra.

Readers who want to know more can consult the last section, *Further Study*, of this paper. It includes a listing of many papers on available on the web.

The American physicist and mathematician David Hestenes initiated the modern development of geometric algebra in the 1960's. He built on the work of Hamilton, Grassmann, and Clifford a century or so earlier. After a slow start, geometric algebra has today attracted many workers in many disciplines.

Hestenes was awarded the 2002 Oersted Medal, the American Association of Physics Teachers "most prestigious award". His medal lecture, "Reforming the Mathematical Language of Physics" [5], was published in The American Journal of Physics, which has published several papers on geometric algebra. In his lecture, Hestenes claims that "geometric algebra simplifies and clarifies the structure of physics, and . . . [thus has] immense implications for physics instruction at all grade levels." I believe that this is equally true of mathematics.

1.2 The Geometric Algebra

The most popular algebraic structure today for Euclidean n-space is the inner product space $\mathbb{R}^n$. This section presents a powerful extension of this structure, the geometric algebra $\mathbb{G}^n$. In subsequent sections, armed with this algebra, we will unify, simplify, and generalize many areas of mathematics, and give several applications.

1.2.1. The geometric algebra $\mathbb{G}^n$. I first present the structure of $\mathbb{G}^n$ concisely, and then, in the next subsection, elaborate.

The geometric algebra $\mathbb{G}^n$ is an extension of the inner product space $\mathbb{R}^n$; every vector in $\mathbb{R}^n$ is also in $\mathbb{G}^n$. First, it is an associative algebra with one. That is, it is a vector space with a product satisfying properties G1-G4 for all scalars a and $A, B, C \in \mathbb{G}^n$:

G1. $A(B+C) = AB + AC$, $(B+C)A = BA + CA$.
G2. $(aA)B = A(aB) = a(AB)$.
G3. $(AB)C = A(BC)$.
G4. $1A = A1 = A$.

The product is called the *geometric product*. Members of $\mathbb{G}^n$ are called *multivectors*. This allows us to reserve the term "vector" for vectors in $\mathbb{R}^n$. (They are also multivectors.) This is a convenient terminology. We list two more properties.

G5. The geometric product of $\mathbb{G}^n$ is linked to the algebraic structure of $\mathbb{R}^n$ by

$$\mathbf{uu} = \mathbf{u} \cdot \mathbf{u} = |\mathbf{u}|^2 \quad \text{for all } \mathbf{u} \in \mathbb{R}^n. \tag{1.1}$$

G6. Every orthonormal basis for $\mathbb{R}^n$ determines a *canonical basis* (defined below) for the vector space $\mathbb{G}^n$.

That's it! That's the geometric algebra. We have not proved that the mathematical structure just described exists. For that, see [9].

1.2.2. Elaboration. Equation (1.1) shows that nonzero vectors have an inverse in $\mathbb{G}^n$: $\mathbf{u}^{-1} = \mathbf{u}/|\mathbf{u}|^2$.

The first step below is a *polarization identity*. You can verify it by multiplying out the dot product on the right side. Then use Eq. (1.1) and distributivity:

$$\begin{aligned} \mathbf{u} \cdot \mathbf{v} &= \frac{1}{2}\big((\mathbf{u}+\mathbf{v}) \cdot (\mathbf{u}+\mathbf{v}) - \mathbf{u} \cdot \mathbf{u} - \mathbf{v} \cdot \mathbf{v}\big) \\ &= \frac{1}{2}\left((\mathbf{u}+\mathbf{v})^2 - \mathbf{u}^2 - \mathbf{v}^2\right) = \frac{1}{2}\left(\mathbf{uv} + \mathbf{vu}\right). \end{aligned}$$

If $\mathbf{u}$ and $\mathbf{v}$ are orthogonal, then this equation gives the important

$$\mathbf{vu} = -\mathbf{uv}. \quad (\mathbf{u}, \mathbf{v} \text{ orthogonal}) \tag{1.2}$$

Example: $(1 + \mathbf{e}_1\mathbf{e}_2)(\mathbf{e}_1 - 2\mathbf{e}_2) = -\mathbf{e}_1 - 3\mathbf{e}_2$.

If $\mathbf{u}$ and $\mathbf{v}$ are orthogonal and nonzero, then from Eq. (1.1), $(\mathbf{uv})^2 = \mathbf{uvuv} = -\mathbf{uuvv} = -|\mathbf{u}|^2|\mathbf{v}|^2 < 0$. Therefore $\mathbf{uv}$ is not a scalar or a vector. It is something new, a 2-*vector*, or *bivector*.

Canonical basis. Let $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4\}$ be an orthonormal basis for $\mathbb{R}^4$. This example of a *canonical basis* for the vector space $\mathbb{G}^4$ suffices to understand the concept:

1	basis for 0-vectors (scalars)
$\mathbf{e}_1 \quad \mathbf{e}_2 \quad \mathbf{e}_3 \quad \mathbf{e}_4$	basis for 1-vectors (vectors)
$\mathbf{e}_1\mathbf{e}_2 \quad \mathbf{e}_1\mathbf{e}_3 \quad \mathbf{e}_1\mathbf{e}_4 \quad \mathbf{e}_2\mathbf{e}_3 \quad \mathbf{e}_2\mathbf{e}_4 \quad \mathbf{e}_3\mathbf{e}_4$	basis for 2-vectors (bivectors)
$\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3 \quad \mathbf{e}_1\mathbf{e}_2\mathbf{e}_4 \quad \mathbf{e}_1\mathbf{e}_3\mathbf{e}_4 \quad \mathbf{e}_2\mathbf{e}_3\mathbf{e}_4$	basis for 3-vectors (trivectors)
$\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3\mathbf{e}_4$	basis for 4-vectors.

The subscripts on the products of $\mathbf{e}$'s are increasing from left to right, and all such products are in the basis.

According to Eq. (1.2), rearranging the order of the $\mathbf{e}$'s in a member of the basis at most changes its sign. Thus the original product and its rearrangement are linearly dependent. We cannot use both in a basis. The canonical basis uses the arrangement with the subscripts increasing from left to right.

Since vectors in $\mathbb{R}^n$ are in $\mathbb{G}^n$ and since $\mathbb{G}^n$ is closed under the geometric product, every linear combination of geometric products of vectors from $\mathbb{R}^n$ is in $\mathbb{G}^n$. A canonical basis shows that *all* multivectors are of this form.

Products of k different $\mathbf{e}$'s span the subspace of *k-vectors*. Properties G1-G6 imply that k-vectors are independent of the orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \dots, \ \mathbf{e}_n\}$.

The (one and only) zero is a k-vector for all k. This is necessary if k-vectors are to form a subspace of $\mathbb{G}^n$.

We might try to form a 5-vector in $\mathbb{G}^4$, e.g., $\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3\mathbf{e}_4\mathbf{e}_2$. But by the product rules, Eqs. (1.1) and (1.2), this is equal to $\mathbf{e}_1\mathbf{e}_3\mathbf{e}_4$. There are no 5-vectors in $\mathbb{G}^4$. More generally, there are no m-vectors in $\mathbb{G}^n$ with $m > n$.

Each member of a canonical basis contains a given $\mathbf{e}$ or it does not. Thus $\mathbb{G}^n$ has dimension 2^n.

How Geometric Algebra Works

Geometric algebra represents geometric objects in $\mathbb{R}^n$ with members of $\mathbb{G}^n$.

Geometric algebra represents geometric operations on these objects with algebraic operations in $\mathbb{G}^n$.

Coordinates are not used in these representations.

Geometric objects include points, lines, planes, circles, and spheres. Geometric operations include rotations, projections, constructions of lines between two points, constructions of circles through three points, determining areas of parallelepipeds, and determining angles between subspaces.

Abbreviate "geometric algebra" to GA and "vector algebra" to VA.

Let's see what we can do with GA.

1.3 The Inner and Outer Products

We investigate the geometric product of two given vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$. Let $\{\mathbf{e}_1, \mathbf{e}_2\}$ be an orthonormal basis for a plane containing the vectors. Let $\mathbf{u} = a\,\mathbf{e}_1 + b\,\mathbf{e}_2$ and $\mathbf{v} = c\,\mathbf{e}_1 + d\,\mathbf{e}_2$. Then from the product rules, Eqs. (1.1) and (1.2),

$$\mathbf{uv} = (ac + bd) \; + \; (ad - bc)\,\mathbf{e}_1\mathbf{e}_2. \tag{1.3}$$

1.3.1. The inner product. The first term on the right side of Eq. (1.3), $ac + bd$, is the usual inner product of $\mathbf{u}$ and $\mathbf{v}$: $\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}|\,|\mathbf{v}| \cos\theta$.

1.3.2. The outer product. The second term on the right side of Eq. (1.3) is the *outer product* of $\mathbf{u}$ and $\mathbf{v}$. This bivector is denoted $\mathbf{u} \wedge \mathbf{v}$. Just as $\mathbf{u}$ represents an oriented length, $\mathbf{u} \wedge \mathbf{v}$ represents an oriented area. For the factor $ad - bc$ is the signed area of the parallelogram with sides $\mathbf{u}$ and $\mathbf{v}$: $|\mathbf{u}|\,|\mathbf{v}| \sin\theta$. And $\mathbf{i} \equiv \mathbf{e}_1\mathbf{e}_2$ specifies the plane in which the area resides. Thus

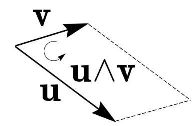


Fig. 1: The bivector $\mathbf{u} \wedge \mathbf{v}$.

$$\mathbf{u} \wedge \mathbf{v} = (ad - bc)\,\mathbf{e}_1\mathbf{e}_2 = |\,\mathbf{u}\,|\,|\,\mathbf{v}\,|\, \sin\theta\,\mathbf{i}\,. \tag{1.4}$$

See Fig. 1. (If you want to know why $\mathbf{e}_1\mathbf{e}_2$ is denoted $\mathbf{i}$, square it. We will have much more to say about this.)

The inner product $\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}|\,|\mathbf{v}| \cos\theta$ is intrinsic to the vectors, not depending on the basis $\{\mathbf{e}_1, \mathbf{e}_2\}$. So too with the outer product. For $|\,\mathbf{u}\,|\,|\,\mathbf{v}|\, \sin\theta$ is intrinsic. And if we rotate to another basis $\begin{pmatrix} \mathbf{e}'_1 \\ \mathbf{e}'_2 \end{pmatrix} = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} \mathbf{e}_1 \\ \mathbf{e}_2 \end{pmatrix}$, then $\mathbf{i}' = \mathbf{e}'_1\mathbf{e}'_2 = \mathbf{i}$.

In 3D VA, the oriented area $\mathbf{u} \wedge \mathbf{v}$ is represented by the cross product $\mathbf{u} \times \mathbf{v}$. But VA does not generalize $\times$ to higher dimensions. Moreover, we shall see that even in 3D, $\wedge$ is superior to $\times$ in several respects. Thus the cross product plays a limited role in GA.

1.3.3. The fundamental identity. Rewrite Eq. (1.3) in terms of the inner and outer products to obtain the *fundamental identity*

$$\mathbf{uv} = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \wedge \mathbf{v}. \tag{1.5}$$

Forget the coordinates of $\mathbf{u}$ and $\mathbf{v}$ which led to this equation. Remember that the geometric product of two vectors is the sum of a scalar and a bivector, both of which have a simple geometric interpretation.

There is no hint in VA that $\cdot$ and $\times$ (reformulated to $\wedge$) are parts of a whole: the geometric product, an associative product in which nonzero vectors have an inverse.[1]

1.3.4. Important miscellaneous facts. We will use them without comment.

- $\mathbf{u} \wedge \mathbf{v}$, unlike $\mathbf{uv}$, is always a bivector.
- $\mathbf{uv}$ is a bivector $\Leftrightarrow$ $\mathbf{u} \cdot \mathbf{v} = 0$ $\Leftrightarrow$ $\mathbf{u} \perp \mathbf{v}$ $\Leftrightarrow$ $\mathbf{uv} = \mathbf{u} \wedge \mathbf{v}$ $\Leftrightarrow$ $\mathbf{uv} = -\mathbf{vu}$.
  In particular, for $i \neq j$, $\mathbf{e}_i\,\mathbf{e}_j = \mathbf{e}_i \wedge \mathbf{e}_j$.
- $\mathbf{uv}$ is a scalar $\Leftrightarrow$ $\mathbf{u} \wedge \mathbf{v} = 0$ $\Leftrightarrow$ $\mathbf{u} \parallel \mathbf{v}$ $\Leftrightarrow$ $\mathbf{uv} = \mathbf{u} \cdot \mathbf{v}$ $\Leftrightarrow$ $\mathbf{uv} = \mathbf{vu}$.
- $\mathbf{v} \wedge \mathbf{u} = -(\mathbf{u} \wedge \mathbf{v})$.
- The inner and outer products are the symmetric and antisymmetric parts of the geometric product: $\mathbf{u} \cdot \mathbf{v} = \frac{1}{2}(\mathbf{uv} + \mathbf{vu})$ and $\mathbf{u} \wedge \mathbf{v} = \frac{1}{2}(\mathbf{uv} - \mathbf{vu})$.

[1] Not every nonzero multivector has an inverse. Example: Let $|\mathbf{u}| = 1$. If $1 - \mathbf{u}$ had an inverse, then right multiply $(1 + \mathbf{u})(1 - \mathbf{u}) = 0$ by the inverse to obtain $1 + \mathbf{u} = 0$, a contradiction.

1.4 Represent Subspaces

A *blade* $\mathbf{B}$ for a k-dimensional subspace of $\mathbb{R}^n$ is a product of members of an orthogonal basis for the subspace: $\mathbf{B} = \mathbf{b}_1\mathbf{b}_2\cdots\mathbf{b}_k$. We call $\mathbf{B}$ a *k-blade*, or a blade of *grade k*. Nonzero scalars are 0–blades.

Geometric algebra represents subspaces with their blades.

A positive multiple of a blade represents the same subspace as the blade. A negative multiple represents the same subspace but with the opposite orientation. For example, $6\mathbf{e}_1\mathbf{e}_3$ and $-\mathbf{e}_1\mathbf{e}_3 = \mathbf{e}_3\mathbf{e}_1$ represent opposite orientations of a 2D subspace.

The inverse of $\mathbf{B}$ is $\mathbf{B}^{-1} = \mathbf{b}_k\cdots\mathbf{b}_2\mathbf{b}_1/|\mathbf{b}_k|^2\cdots|\mathbf{b}_2|^2|\mathbf{b}_1|^2$.

We use **bold** to designate blades, with lower case reserved for vectors. (Exception: $\mathbf{i}$.) We use upper case *italic* $(A, B, C, \ldots)$ to denote general multivectors.

We shorten "the subspace represented by blade $\mathbf{B}$" to "the subspace $\mathbf{B}$", or simply "$\mathbf{B}$". For example, set theoretic relationships between blades, e.g., "$\mathbf{A} \subseteq \mathbf{B}$", refer to the subspaces they represent.

1.4.1. The pseudoscalar. The (unit) *pseudoscalar* $\mathbf{I} = \mathbf{e}_1\,\mathbf{e}_2\,\cdots\,\mathbf{e}_n$ is an important member of $\mathbb{G}^n$. (We use lower case in 2D: $\mathbf{i} = \mathbf{e}_1\mathbf{e}_2$, as in Sec. 1.3.2.) Since n-vectors form a 1-dimensional subspace of $\mathbb{G}^n$, every n-vector is a scalar multiple of $\mathbf{I}$. If $\mathbf{e}'_1\,\mathbf{e}'_2\,\cdots\,\mathbf{e}'_n = \mathbf{I}'$ for another orthonormal basis, then $\mathbf{I}'$ is a unit n-vector. Thus $\mathbf{I}' = \pm\mathbf{I}$. We say that the orthonormal bases have the *same orientation* if $\mathbf{I}' = \mathbf{I}$ and the *opposite orientation* if $\mathbf{I}' = -\mathbf{I}$. We saw the 2D case in Sec. 1.3.2.

Note that $\mathbf{I}^{-1} = \mathbf{e}_n\cdots\mathbf{e}_2\mathbf{e}_1$. Starting with $\mathbf{I} = \mathbf{e}_1\,\mathbf{e}_2\,\cdots\,\mathbf{e}_n$, move $\mathbf{e}_n$ to the left with $n-1$ adjacent swaps, then move $\mathbf{e}_{n-1}$ next to it with $n-2$ adjacent swaps, and so on until $\mathbf{I}^{-1}$ is obtained. This requires $(n-1)n/2$ swaps in all. Thus from Eq. (1.2), $\mathbf{I}^{-1} = (-1)^{(n-1)n/2}\,\mathbf{I} = \pm\mathbf{I}$.

All blades representing a subspace are scalar multiples of each other. For they are the pseudoscalars of the geometric algebra of the subspace.

1.4.2. Duality. Define the *dual* of a multivector A:

$$A^* = A/\mathbf{I}\ (\equiv A\mathbf{I}^{-1})\,. \tag{1.6}$$

1.4.3. Theorem. (Orthogonal complement) If $\mathbf{A}$ is a j-blade, then $\mathbf{A}^*$ is an $(n-j)$-blade representing the orthogonal complement of $\mathbf{A}$.

Proof. Extend an orthonormal basis $\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_j\}$ of $\mathbf{A}$ to an orthonormal basis $\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_j, \mathbf{a}_{j+1}, \ldots, \mathbf{a}_n\}$ of $\mathbb{R}^n$ Then

$$\mathbf{A}^* = \pm\mathbf{A}\mathbf{I} = \pm(\mathbf{a}_1\mathbf{a}_2\cdots\mathbf{a}_j)\,(\mathbf{a}_1\mathbf{a}_2\cdots\mathbf{a}_n) = \pm\mathbf{a}_{j+1}\cdots\mathbf{a}_n,$$

which represents the orthogonal complement of $\mathbf{A}$. □

The $*$ operator of GA is the $\perp$ operator of VA. Because VA does not represent general subspaces, it cannot implement $\perp$ as an algebraic operation.

1.4.4. Corollary. The $(n-1)$-vectors in $\mathbb{G}^n$ are blades.

Proof. From a canonical basis, an $(n-1)$-vector P is a sum of $(n-1)$-blades. From the theorem, P^* is a sum of vectors, which is a vector, which is a 1-blade. From the theorem again, $P^{**} = \pm P$ is an $(n-1)$-blade. □

In $\mathbb{R}^3$ the vectors $(\mathbf{u}\wedge\mathbf{v})^*$ and $\mathbf{u}\times\mathbf{v}$ are parallel, as both are orthogonal to the oriented plane $\mathbf{u}\wedge\mathbf{v}$. From Eq. (1.4), $\mathbf{u}\times\mathbf{v} = (\mathbf{u}\wedge\mathbf{v})^*$.

1.5 Extend the Inner and Outer Products

We extend the products from vectors to all multivectors.

Let $\langle\, C\,\rangle_j$ denote the j-vector part of the multivector C. For example, from the fundamental identity Eq. (1.5), $\langle\, \mathbf{ab}\,\rangle_0 = \mathbf{a}\cdot\mathbf{b}$, $\langle\, \mathbf{ab}\,\rangle_1 = 0$, and $\langle\, \mathbf{ab}\,\rangle_2 = \mathbf{a}\wedge\mathbf{b}$.

1.5.1. Inner product. Let A be a j-vector and B a k-vector. Define:[2]

$$A\cdot B = \langle\, AB\,\rangle_{k-j}\,. \tag{1.7}$$

(Thus if $j > k$, then $A\cdot B = 0$.) Examples:

$$\begin{aligned}(\mathbf{e}_1\mathbf{e}_2)\cdot(\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3) &= \langle\, \mathbf{e}_1\mathbf{e}_2\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3\,\rangle_{3-2} = -\mathbf{e}_3\,,\\ (\mathbf{e}_1)\cdot(\mathbf{e}_2\mathbf{e}_3) &= \langle\, \mathbf{e}_1\mathbf{e}_2\mathbf{e}_3\,\rangle_{2-1} = 0\,.\end{aligned} \tag{1.8}$$

1.5.2. Outer product. Define:

$$A\wedge B = \langle\, AB\,\rangle_{j+k}\,. \tag{1.9}$$

(Thus if $j + k > n$, then $\mathbf{A}\wedge\mathbf{B} = 0$.) Examples:

$$\begin{aligned}(\mathbf{e}_1\mathbf{e}_2)\wedge(\mathbf{e}_3) &= \langle\, \mathbf{e}_1\mathbf{e}_2\mathbf{e}_3\,\rangle_{2+1} = \mathbf{e}_1\mathbf{e}_2\mathbf{e}_3,\\ (\mathbf{e}_1\mathbf{e}_2)\wedge(\mathbf{e}_2\mathbf{e}_3) &= \langle\, \mathbf{e}_1\mathbf{e}_2\mathbf{e}_2\mathbf{e}_3\,\rangle_{2+2} = 0\,.\end{aligned} \tag{1.10}$$

Extend the inner and outer products to multivectors of mixed grades by distributivity.

Here are some properties of the geometric, inner, and outer products.

1.5.3. Theorem. ($\mathbf{AB}$) If $\mathbf{A}$ is a j-blade and $\mathbf{B}$ a k-blade, then the possible grades in $\mathbf{AB}$ are $|k-j|$, $|k-j|+2$, $\ldots$, $k+j-2$, $k+j$.

1.5.4. Theorem. ($\mathbf{A}\cdot\mathbf{B}$)
a. $\mathbf{A}\cdot\mathbf{B}$ is a blade, or zero.
b. If $\mathbf{A}\cdot\mathbf{B}\neq 0$, then $\mathbf{A}\cdot\mathbf{B}\subseteq\mathbf{B}$.
c. If $\mathbf{A}\subseteq\mathbf{B}$, then $\mathbf{A}\cdot\mathbf{B} = \mathbf{AB}$.
d. For every $\mathbf{a}\in\mathbf{A}$, $\mathbf{a}\perp\mathbf{A}\cdot\mathbf{B}$.
e. $|\mathbf{A}\cdot\mathbf{A}| = |\mathbf{A}|^2$.

1.5.5. Theorem. ($\mathbf{A}\wedge\mathbf{B}$)
a. $\mathbf{A}\wedge\mathbf{B}$ is a blade, or zero.
b.

$$\mathbf{A}\wedge\mathbf{B} = \begin{cases}\operatorname{span}(\mathbf{A},\mathbf{B}), & \text{if } \mathbf{A}\cap\mathbf{B} = \{0\};\\ 0, & \text{if } \mathbf{A}\cap\mathbf{B}\neq\{0\}.\end{cases} \tag{1.11}$$

c. The outer product is associative. (The cross product is not.)
d. $\mathbf{A}\wedge\mathbf{B} = (-1)^{jk}(\mathbf{B}\wedge\mathbf{A})$. (This generalizes $\mathbf{a}\wedge\mathbf{b} = -(\mathbf{b}\wedge\mathbf{a})$.)

[2] $|k-j|$ is used more often than $k-j$ in the definition of the inner product, Eq. (1.7) [1]. Dorst has argued the advantages of the inner product as defined here, which is often called the left contraction, and denoted "$\rfloor$". [2].

1.5.6. Duality. The inner and outer products are dual on blades:

$$(\mathbf{A}\cdot\mathbf{B})^* = \mathbf{A}\wedge\mathbf{B}^*, \quad (\mathbf{A}\wedge\mathbf{B})^* = \mathbf{A}\cdot\mathbf{B}^*. \tag{1.12}$$

To see, e.g., the second equality, let $\mathbf{A}$ be a j-blade and $\mathbf{B}$ be a k-blade. Then

$$\mathbf{A}\cdot\mathbf{B}^* = \langle\,\mathbf{A}(\mathbf{B}\mathbf{I}^{-1})\,\rangle_{(n-k)-j} = \langle\,(\mathbf{A}\mathbf{B})\mathbf{I}^{-1}\,\rangle_{n-(j+k)} = \langle\,\mathbf{A}\mathbf{B}\,\rangle_{j+k}\,\mathbf{I}^{-1} = (\mathbf{A}\wedge\mathbf{B})^*.$$

1.5.7. Lemma.. Let $\mathbf{a}$ be a vector and $\mathbf{B}$ a blade. Decompose $\mathbf{a}$ with respect to $\mathbf{B}$: $\mathbf{a} = \mathbf{a}_\| + \mathbf{a}_\perp$, where $\mathbf{a}_\| \in \mathbf{B}$, $\mathbf{a}_\perp \perp \mathbf{B}$. See Fig. 2.

Suppose that $\mathbf{B} = \mathbf{b}_1\mathbf{b}_2\cdots\mathbf{b}_k$ be a k-blade. Then:

a. If $\mathbf{a}_\| \in \mathbf{B}$, then $\mathbf{a}_\|\cdot\mathbf{B} = \mathbf{a}_\|\mathbf{B}$ and $\mathbf{a}_\|\wedge\mathbf{B} = 0$.
Also, $\mathbf{a}_\|\cdot\mathbf{B}$ is a $(k-1)$-blade in $\mathbf{B}$.

b. If $\mathbf{a}_\perp \perp \mathbf{B}$, then $\mathbf{a}_\perp\wedge\mathbf{B} = \mathbf{a}_\perp\mathbf{B}$ and $\mathbf{a}_\perp\cdot\mathbf{B} = 0$.
Also, $\mathbf{a}_\perp\wedge\mathbf{B}$ is a $(k+1)$-blade representing $\operatorname{span}(\mathbf{a}_\perp, \mathbf{B})$.

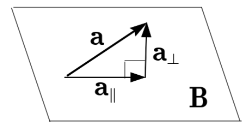


Fig. 2: Projection & rejection.

1.5.8. Theorem. (Extend the fundamental identity.) Let $\mathbf{B}$ be a k-blade. Then for every vector $\mathbf{a}$,

$$\mathbf{a}\mathbf{B} = \mathbf{a}\cdot\mathbf{B} + \mathbf{a}\wedge\mathbf{B}. \tag{1.13}$$

In this equation,

$\mathbf{a}\cdot\mathbf{B}$ is a $(k-1)$-blade in $\mathbf{B}$ (or $\mathbf{0}$),
$\mathbf{a}\wedge\mathbf{B}$ is a $(k+1)$-blade representing $\operatorname{span}(\mathbf{a}, \mathbf{B})$ (or $\mathbf{0}$).

Proof. Using the lemma four times in Step (3),

$$\begin{aligned}\mathbf{a}\mathbf{B} &= (\mathbf{a}_\| + \mathbf{a}_\perp)\mathbf{B} = \mathbf{a}_\|\mathbf{B} + \mathbf{a}_\perp\mathbf{B} = \mathbf{a}_\|\cdot\mathbf{B} + \mathbf{a}_\perp\cdot\mathbf{B} + \mathbf{a}_\perp\wedge\mathbf{B} + \mathbf{a}_\|\wedge\mathbf{B} \\ &= (\mathbf{a}_\| + \mathbf{a}_\perp)\cdot\mathbf{B} + (\mathbf{a}_\perp + \mathbf{a}_\|)\wedge\mathbf{B} = \mathbf{a}\cdot\mathbf{B} + \mathbf{a}\wedge\mathbf{B}.\end{aligned}$$

For the rest of the theorem, apply the lemma to $\mathbf{a}\cdot\mathbf{B} = \mathbf{a}_\|\cdot\mathbf{B}$ and $\mathbf{a}\wedge\mathbf{B} = \mathbf{a}_\perp\wedge\mathbf{B}$. □

In general, $\mathbf{A}\mathbf{B} \neq \mathbf{A}\cdot\mathbf{B} + \mathbf{A}\wedge\mathbf{B}$. Example: $\mathbf{A} = \mathbf{e}_1\mathbf{e}_2$ and $\mathbf{B} = \mathbf{e}_2\mathbf{e}_3$.

1.5.9. Theorem. (Projections and rejections.) Let $\mathbf{a}$ be a vector and $\mathbf{B}$ a blade. Decompose $\mathbf{a}$ with respect to $\mathbf{B}$: $\mathbf{a} = \mathbf{a}_{\parallel} + \mathbf{a}_{\perp}$. Then

$$\mathbf{a}_{\parallel} = (\mathbf{a} \cdot \mathbf{B})/\mathbf{B}, \tag{1.14}$$
$$\mathbf{a}_{\perp} = (\mathbf{a} \wedge \mathbf{B})/\mathbf{B}. \tag{1.15}$$

Proof. From the lemma, $\mathbf{a}_{\perp}\mathbf{B} = \mathbf{a}_{\perp} \wedge \mathbf{B} + \mathbf{a}_{\parallel} \wedge \mathbf{B} = \mathbf{a} \wedge \mathbf{B}$. This gives Eq. (1.15). $\square$

VA has no analogs of these simple formulas, except when $\mathbf{B} = \mathbf{b}$, a vector, in the projection formula Eq. (1.14).

1.5.10. Theorem. (Subspace membership test.) For every vector $\mathbf{a}$ and blade $\mathbf{B}$,

$$\mathbf{a} \in \mathbf{B} \;\Leftrightarrow\; \mathbf{a} \wedge \mathbf{B} = 0, \tag{1.16}$$
$$\mathbf{a} \in \mathbf{B} \;\Leftrightarrow\; \mathbf{a} \cdot \mathbf{B}^* = 0. \tag{1.17}$$

Proof. From $\mathbf{a} = \mathbf{a}_{\parallel} + \mathbf{a}_{\perp}$, $\mathbf{a} \in \mathbf{B} \;\Leftrightarrow\; \mathbf{a}_{\perp} = 0$. This, with Eq. (1.15), gives Eq. (1.16).
From $(\mathbf{a} \cdot \mathbf{B})^* = \mathbf{a} \wedge \mathbf{B}^*$, Eq. (1.12), applied to $\mathbf{B}^*$, $\mathbf{a} \cdot \mathbf{B}^* = 0 \;\Leftrightarrow\; \mathbf{a} \wedge \mathbf{B} = 0$. This, with Eq. (1.16), gives Eq. (1.17). $\square$

We say that $\mathbf{B}$ is a *direct representation* of the subspace and $\mathbf{B}^*$ is a *dual representation*. Both are useful.

The distance from the endpoint of $\mathbf{a}$ to $\mathbf{B}$ is $|\,\mathbf{a}_{\perp}| = |\,(\mathbf{a} \wedge \mathbf{B})/\mathbf{B}\,|$. If $\mathbf{B}$ is a hyperplane, e.g., a plane in $\mathbb{R}^3$, then it divides $\mathbb{R}^n$ into two sides. Then $\mathbf{a} \wedge \mathbf{B}$ is an n-vector. The scalar $(\mathbf{a} \wedge \mathbf{B})^*/\,|\mathbf{B}|$ is a *signed* distance from the endpoint of $\mathbf{a}$ to $\mathbf{B}$.

1.5.11. Theorem. (Reflections.) Let $\mathsf{F}_{\mathbf{B}}(\mathbf{a})$ be the reflection of a vector $\mathbf{a}$ in a blade $\mathbf{B}$

a. If $\mathbf{B}$ is a k-blade, then $\mathsf{F}_{\mathbf{B}}(\mathbf{a}) = (-1)^{k+1}\mathbf{B}\mathbf{a}\mathbf{B}^{-1}$.
b. If $\mathbf{B}$ is a hyperplane and $\mathbf{b} = \mathbf{B}^*$ is a vector normal to $\mathbf{B}$, then $\mathsf{F}_{\mathbf{B}}(\mathbf{a}) = -\mathbf{b}\mathbf{a}\mathbf{b}^{-1}$.

Proof. a. We prove only the case $\mathbf{B} = \mathbf{b}$, a vector. From Figure 2, $\mathsf{F}_{\mathbf{B}}(\mathbf{a}) = \mathbf{a}_{\parallel} - \mathbf{a}_{\perp}$. Thus from Eqs. (1.14) and (1.15),

$$\mathsf{F}_{\mathbf{B}}(\mathbf{a}) = \mathbf{a}_{\parallel} - \mathbf{a}_{\perp} = (\mathbf{a} \cdot \mathbf{b})/\mathbf{b} - (\mathbf{a} \wedge \mathbf{b})/\mathbf{b} = (\mathbf{b} \cdot \mathbf{a})/\mathbf{b} + (\mathbf{b} \wedge \mathbf{a})/\mathbf{b} = \mathbf{b}\mathbf{a}\mathbf{b}^{-1}. \tag{1.18}$$

b. Setting $\mathbf{B} = \mathbf{b}\mathbf{I}$ in Part (a),

$$\mathsf{F}_{\mathbf{B}}(\mathbf{a}) = (-1)^n(\mathbf{b}\mathbf{I})\mathbf{a}(\mathbf{b}\mathbf{I}^{-1}) = (-1)^n\mathbf{b}(-1)^{n-1}\mathbf{a}\mathbf{I}\,\mathbf{I}^{-1}\mathbf{b}^{-1} = -\mathbf{b}\mathbf{a}\mathbf{b}^{-1}. \tag{1.19}$$

$\square$

1.6 k-Volumes

1.6.1. k-volumes. Let the blade $\mathbf{B} = \mathbf{b}_1\mathbf{b}_2\cdots\mathbf{b}_k$ be a product of nonzero orthogonal vectors. The *norm* of $\mathbf{B}$ is $|\mathbf{B}| = |\mathbf{b}_1||\mathbf{b}_2|\cdots|\mathbf{b}_k|$. This is the k-volume of the rectangular parallelepiped spanned by the $\mathbf{b}$'s. In this way $\mathbf{B}$ represents both a subspace and a k-volume in the subspace. As $\mathbf{B}$ has neither a shape nor a position, neither does the k-volume. For example, $\mathbf{b}_1\mathbf{b}_2 = (2\mathbf{b}_1)(\frac{1}{2}\mathbf{b}_2)$.

The projection formula Eq. (1.14) generalizes: the projection of $\mathbf{A}$ onto $\mathbf{B}$ is $P_{\mathbf{B}}(\mathbf{A}) = (\mathbf{A}\cdot\mathbf{B})/\mathbf{B}$.[3] The angle θ between $\mathbf{A}$ and $\mathbf{B}$ is given by a ratio of j-volumes (j = grade($\mathbf{A}$)): $\cos\theta = |P_{\mathbf{B}}(\mathbf{A})|/|\mathbf{A}| = |\mathbf{A}\cdot\mathbf{B}|/|\mathbf{A}|\,|\mathbf{B}|$.[4] For example, the angle between the line $\mathbf{A} = \mathbf{e}_2 + \mathbf{e}_3$ and the plane $\mathbf{B} = \mathbf{e}_1\mathbf{e}_2$ is $\pi/4$.

1.6.2. Theorem. Let $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k$ be linearly independent vectors. Then $\mathbf{B} = \mathbf{u}_1 \wedge \mathbf{u}_2 \wedge \cdots \wedge \mathbf{u}_k$ is a k-blade.

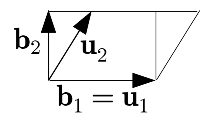


Fig. 3: $\mathbf{u}_1\wedge\mathbf{u}_2 = \mathbf{b}_1\mathbf{b}_2$.

Proof. By induction. We perform a GA version of Gram-Schmidt orthogonalization on the $\mathbf{u}_i$. Note how the rejection operator of GA simplifies the VA version.

The $k = 1$ case is clear. For ease of notation, we induct from $k = 2$ to $k = 3$. Thus we assume that $\mathbf{u}_1 \wedge \mathbf{u}_2$ is a 2-blade, i.e., $\mathbf{u}_1 \wedge \mathbf{u}_2 = \mathbf{b}_1\,\mathbf{b}_2$ with $\mathbf{b}_1 \perp \mathbf{b}_2$. See Fig. 3.

Let $\mathbf{b}_3 = \{\mathbf{u}_3 \wedge (\mathbf{u}_1 \wedge \mathbf{u}_2)\}/(\mathbf{u}_1 \wedge \mathbf{u}_2)$ be the rejection (Eq. (1.15)) of $\mathbf{u}_3$ by the subspace $\mathbf{u}_1 \wedge \mathbf{u}_2$.[5] Since $\mathbf{u}_1, \mathbf{u}_2$, and $\mathbf{u}_3$ are linearly independent, $\mathbf{b}_3 \neq 0$. We have

$$\mathbf{u}_1 \wedge \mathbf{u}_2 \wedge \mathbf{u}_3 = (\mathbf{u}_1 \wedge \mathbf{u}_2) \wedge \mathbf{b}_3 = (\mathbf{b}_1\mathbf{b}_2) \wedge \mathbf{b}_3 = \mathbf{b}_1\mathbf{b}_2\,\mathbf{b}_3, \tag{1.20}$$

a 3-blade. The last equality follows from the definition of $\wedge$, Eq. (1.9). □

We now have two ways to express a blade: as a geometric product of nonzero orthogonal vectors and an outer product of linearly independent vectors.

We are beginning to see some of the power of GA promised in the introduction. Geometric objects are represented directly as members of the algebra. Examples: Oriented subspaces and k-volumes are represented by blades. Geometric relationships are expressed algebraically. Examples: $\mathbf{x} \in \mathbf{B} \Leftrightarrow \mathbf{x} \wedge \mathbf{B} = 0$, $\mathbf{x} \perp \mathbf{B} \Leftrightarrow \mathbf{x}\cdot\mathbf{B} = 0$. Geometric operations on objects are represented by algebraic operations. Examples: $\mathbf{A}^* = \mathbf{A}/\mathbf{I}$, $\mathbf{a}_\perp = (\mathbf{a} \wedge \mathbf{B})/\mathbf{B}$, $\mathbf{a}_\| = (\mathbf{a}\cdot\mathbf{B})/\mathbf{B}$. The result of an operation can be substituted in other expressions, which can then be manipulated algebraically.

All this without using coordinates. Geometric algebra is *coordinate-free*: coordinates are needed only when specific objects or operations are under consideration.

These features of GA fit naturally into the modern techniques of object oriented computer programming.

[3] Note, for example, that $P_{\mathbf{e}_1\mathbf{e}_2}(\mathbf{e}_2\mathbf{e}_3) = 0$. This tells us that areas in the plane $\mathbf{e}_2\mathbf{e}_3$ projected to the plane $\mathbf{e}_1\mathbf{e}_2$ have area 0 there.

[4] $|(\mathbf{A}\cdot\mathbf{B})/\mathbf{B}| = |\mathbf{A}\cdot\mathbf{B}|/|\mathbf{B}|$ because $\mathbf{B}^{-1}$ is a blade.

[5] To compute $\mathbf{b}_3$, use $(\mathbf{u}_1 \wedge \mathbf{u}_2)^{-1} = (\mathbf{b}_1\mathbf{b}_2)^{-1} = \mathbf{b}_2^{-1}\mathbf{b}_1^{-1} = (\mathbf{b}_2/|\mathbf{b}_2|^2)(\mathbf{b}_1/|\mathbf{b}_1|^2)$.

2 Algebra

2.1 Complex Numbers

2.1.1. Complex numbers. Let $\mathbf{i}$ be the pseudoscalar of a plane in $\mathbb{R}^n$. Then $a + b\mathbf{i}$, scalar + bivector, is a *complex number*. Since $\mathbf{i}^2 = -1$, GA complex numbers are isomorphic to the usual complex numbers. But GA complex numbers are not represented as points in a plane or as 2D vectors.

Let θ be the angle between vectors $\mathbf{u}$ and $\mathbf{v}$. The fundamental identity, Eq. (1.5), shows that the product of two vectors is a complex number:

$$\mathbf{uv} = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \wedge \mathbf{v} = |\mathbf{u}|\,|\mathbf{v}| \cos\theta + \mathbf{i}\,|\mathbf{u}|\,|\mathbf{v}| \sin\theta.$$

Define $\mathrm{e}^{\mathbf{i}\,\theta} = \cos\theta + \mathbf{i}\sin\theta$. Write the complex number $\mathbf{uv}$ in *polar form*:

$$\mathbf{uv} = |\mathbf{u}|\,|\mathbf{v}|\,\mathrm{e}^{\mathbf{i}\,\theta} = r\,\mathrm{e}^{\mathbf{i}\,\theta}. \tag{2.1}$$

Every complex number $a + \mathbf{i}b$ can be put in the polar form $r\mathrm{e}^{\mathbf{i}\,\theta}$ by setting $r = \sqrt{a^2 + b^2}$, $\cos\theta = a/r$, and $\sin\theta = b/r$. Note the familiar $\mathrm{e}^{\mathbf{i}\,\pi/2} = \mathbf{i}$.

Traditional complex numbers use geometrically irrelevant real and imaginary axes. Introducing them breaks the rotational symmetry of the plane. This makes it impossible to implement traditional complex numbers coherently in different planes in higher dimensions.

With GA complex numbers, the a and b in $a + b\mathbf{i}$ and the θ in $\mathrm{e}^{\mathbf{i}\theta}$ have a geometric meaning independent of any particular coordinate system. This is different from traditional complex numbers, where, for example, θ is an angle with respect to the real axis.

The usual complex number i is not needed. It is not part of geometric algebra.

2.1.2. Theorem. The set of complex numbers in $\mathbb{G}^3$ is a subalgebra of $\mathbb{G}^3$.

Proof. Let $\{\mathbf{i}_1 = \mathbf{e}_3\mathbf{e}_2,\ \mathbf{i}_2 = \mathbf{e}_1\mathbf{e}_3,\ \mathbf{i}_3 = \mathbf{e}_2\mathbf{e}_1\}$ be a basis for bivectors in $\mathbb{G}^3$. You can check that

$$\mathbf{i}_1^2 = \mathbf{i}_2^2 = \mathbf{i}_3^2 = -1 \text{ and } \mathbf{i}_1\mathbf{i}_2 = \mathbf{i}_3,\ \mathbf{i}_2\mathbf{i}_3 = \mathbf{i}_1,\ \mathbf{i}_3\mathbf{i}_1 = \mathbf{i}_2. \tag{2.2}$$

Every complex number $a + b\mathbf{i}$ in $\mathbb{G}^3$ can be put in the form $a + b_1\mathbf{i}_1 + b_2\mathbf{i}_2 + b_3\mathbf{i}_3$. From Eq. (2.2), the product of such multivectors is another. By Corollary 1.4.4, the product is a complex number. This is sufficient to prove the theorem. $\square$

The theorem fails in $\mathbb{G}^4$: $(\mathbf{e}_1\mathbf{e}_2)(\mathbf{e}_3\mathbf{e}_4)$ is not a complex number.

The identities in Eq. (2.2) characterize the quaternions. Thus the complex numbers in $\mathbb{G}^3$ form the quaternion algebra.

Traditionally, quaternions have been considered as scalar + vector. So considered, they have not been satisfactorily united with vectors in a common mathematical system [3, 12]. Considered here as scalar + bivector, they are united with vectors in GA.

For many, quaternions are a 19^{th} century mathematical curiosity. But many roboticists, aerospace engineers, and gamers know better: quaternions are the best way to represent rotations in 3D, as we will see in Sec. 2.2.1.

Complex numbers and quaternions are only two of the many algebraic systems embedded in GA. We shall see several more examples. Having these systems embedded in a common structure reveals and clarifies relationships between them. For example, we have seen that the product of two vectors is a complex number.

The exterior (Grassmann) algebra is another system embedded in GA. It consists of outer products of nonzero vectors. It is thus only part of GA, not using the inner or geometric products. Taking advantage of these products simplifies the exterior algebra. For example, the GA dual, whose definition uses the geometric product, is the Hodge star dual, up to a sign. The GA definition allows easier manipulation.

2.2 Rotations

2.2.1. Rotations in $\mathbb{R}^3$. Orthogonal matrices and Euler angles are among the many representations of rotations in $\mathbb{R}^3$ in use today. GA provides a better representation.

In GA the rotation of a vector $\mathbf{u}$ by an angle θ around an axis $\mathbf{n}$ is given by

$$\mathsf{R}_{\mathbf{i}\theta}(\mathbf{u}) = \mathrm{e}^{-\mathbf{nI}\theta/2}\mathbf{u}\mathrm{e}^{\mathbf{nI}\theta/2}. \tag{2.3}$$

We prove this in Sec. 2.2.3. The formula expresses the rotation simply and directly in terms of θ and $\mathbf{n}$. We write $\mathsf{R}_{\mathbf{i}\theta}(\mathbf{u}) = R\mathbf{u}R^{-1}$, where the unit complex number $R = \mathrm{e}^{-\mathbf{nI}\theta/2}$ *represents* the rotation.

Thus in $\mathbb{R}^3$, unit quaternions represent rotations.

2.2.2. Rotations compose simply. Follow the rotation R_1 with the rotation R_2. Then their composition is represented by the product $R = R_2R_1$:

$$M \to R_2(R_1MR_1^{-1})R_2^{-1} = RMR^{-1}.$$

Since the product of unit quaternions is another, we have a simple proof of the important fact that the composition of 3D rotations is a rotation.

As an example, a rotation by 90° around the $\mathbf{e}_3$ axis followed by a rotation of 90° around the $\mathbf{e}_1$ axis is a rotation of 120° around $\mathbf{e}_1 - \mathbf{e}_2 + \mathbf{e}_3$:

$$\begin{aligned}(\cos 45^\circ - \mathbf{e}_1\mathbf{I}\sin 45^\circ)\,(\cos 45^\circ - \mathbf{e}_3\mathbf{I}\sin 45^\circ) \\ = \cos 60^\circ - \frac{\mathbf{e}_1 - \mathbf{e}_2 + \mathbf{e}_3}{\sqrt{3}}\,\mathbf{I}\sin 60^\circ.\end{aligned}$$

In the general case, the composite quaternion $\mathbf{Q} = \cos(\theta/2) + \mathbf{nI}\sin(\theta/2)$, where $\mathbf{n}$ is a unit vector, the axis of the composite rotation.

Quaternions are superior to orthogonal matrices in representing 3D rotations: (i) It is easier to determine the quaternion representation of a rotation than the matrix representation,[6] (ii) It is more efficient to multiply quaternions than matrices, and (iii) If a quaternion product is not quite normalized due to rounding errors, then divide by its norm to make it so; if a product of orthogonal matrices is not orthogonal, then use Gram-Schmidt orthonormalization, which is expensive and not canonical.

[6] *Rodrigues' rotation formula* gives the matrix representation of a 3D rotation [11].

2.2.3. Rotations in $\mathbb{R}^n$. In GA, an *angle* is a bivector $\mathbf{i}\theta$: $\mathbf{i}$ specifies the plane in which it resides and θ specifies its size.[7]

In $\mathbb{R}^n$, an angle $\mathbf{i}\theta$ specifies a rotation: $\mathbf{i}$ specifies the plane of the rotation and θ specifies the amount of rotation. (Only in 3D does a rotation have an axis, the unique direction normal to the plane of the rotation.)

Let the rotation carry the vector $\mathbf{u}$ to the vector $\mathbf{v}$. First suppose that $\mathbf{u} \in \mathbf{i}$. Left multiply $\mathbf{uv} = |\mathbf{u}||\mathbf{v}|\, e^{\mathbf{i}\theta}$, Eq. (2.1), by $\mathbf{u}$ and use $\mathbf{u}^2 = |\mathbf{u}|^2 = |\mathbf{u}|\,|\mathbf{v}|$ to obtain $\mathbf{v} = \mathbf{u}\, e^{\mathbf{i}\,\theta}$; $e^{\mathbf{i}\,\theta}$ rotates $\mathbf{u}$ to $\mathbf{v}$. The analog in standard complex algebra is familiar.

Now consider a general $\mathbf{u}$. Decompose $\mathbf{u}$ with respect to $\mathbf{i}$: $\mathbf{u} = \mathbf{u}_\perp + \mathbf{u}_\parallel$ ($\mathbf{u}_\perp \perp \mathbf{i}$, $\mathbf{u}_\parallel \in \mathbf{i}$). The rotation rotates $\mathbf{u}_\parallel$ as above but does not affect $\mathbf{u}_\perp$. Thus

$$\begin{aligned}
\mathbf{v} &= \mathbf{u}_\perp + \mathbf{u}_\parallel\, e^{\mathbf{i}\,\theta} \\
&= \mathbf{u}_\perp e^{-\mathbf{i}\,\theta/2}\, e^{\mathbf{i}\,\theta/2} + \mathbf{u}_\parallel\, e^{\mathbf{i}\,\theta/2}\, e^{\mathbf{i}\,\theta/2} \\
&= e^{-\mathbf{i}\,\theta/2}\, \mathbf{u}_\perp\, e^{\mathbf{i}\,\theta/2} + e^{-\mathbf{i}\,\theta/2}\, \mathbf{u}_\parallel\, e^{\mathbf{i}\,\theta/2} \qquad (\text{since } \mathbf{u}_\perp \mathbf{i} = \mathbf{i}\mathbf{u}_\perp \text{ and } \mathbf{u}_\parallel \mathbf{i} = -\mathbf{i}\mathbf{u}_\parallel) \\
&= e^{-\mathbf{i}\,\theta/2}\, \mathbf{u}\, e^{\mathbf{i}\,\theta/2}.
\end{aligned} \tag{2.4}$$

In 3D this reduces to Eq. (2.3). For in $\mathbb{G}^3$, $\mathbf{n} = \mathbf{i}^* = \mathbf{i}/\mathbf{I}$.

Rotate rotations. Let R_1 represent a rotation by angle $\mathbf{i}_1\theta_1$. Rotate the rotation: rotate the plane $\mathbf{i}_1$ of this rotation with a rotation represented by R_2. It is easy to show that the rotated rotation is represented by $R_2 R_1 R_2^{-1}$. Rotations rotate just as vectors do!

2.2.4. The special orthogonal group. In nD, $n > 3$, rotations by angles $\mathbf{i}\theta$ are not closed under composition. They generate the orientation and inner product preserving special orthogonal group $\mathsf{SO}(n)$. Given $e^{-\mathbf{i}\,\theta/2}$, choose, using Eq. (2.1), unit vectors $\mathbf{b}_1$ and $\mathbf{b}_2$ so that $e^{-\mathbf{i}\,\theta/2} = \mathbf{b}_1\mathbf{b}_2$. Then the rotation Eq. (2.4) can be written $\mathbf{u} \to (\mathbf{b}_1\mathbf{b}_2)\mathbf{u}(\mathbf{b}_1\mathbf{b}_2)^{-1}$. We may drop the normalization $|\mathbf{b}_i| = 1$ in this formula. Thus a member of $\mathsf{SO}(n)$ can be represented by a product of an even number of nonzero vectors $B = \mathbf{b}_1\mathbf{b}_2 \cdots \mathbf{b}_{2k}$: $\mathbf{u} \to B\mathbf{u}B^{-1}$ (Sec. 2.2.2).

Eq. (2.4) represents a rotation of 2π by $e^{-\mathbf{i}\,2\pi/2} = -1$. The products B above represent the simply connected double covering group of $\mathsf{SO}(n)$, with $\pm B$ representing the same element of $\mathsf{SO}(n)$. This is another algebraic system embedded in GA. Matrices do not represent the double cover.

> "A particular area where geometric algebra provides a unifying language is in the description of rotations. The most fundamental modern treatments such as those of Abraham, Marsden, Arnol'd and Simo use differential topology and describe rotations in terms of the Lie group $\mathsf{SO}(3)$. A rotation is thus an element of a differentiable manifold, and combinations of rotations are described using the group action. Infinitesimal rotations and rotational velocities live in the tangent bundle $\mathsf{TSO}(3)$, a differentiable manifold with distinctly non-trivial topology, from where they can be transported to the tangent space at the identity, identifiable with the Lie algebra $\mathsf{so}(3)$. As throughout all of the differentiable topology formulation of mechanics, a proliferation of manifolds occurs. ... In geometric algebra there is no such proliferation of manifolds: the mathematical arena consists only of elements of the algebra and nothing more." [10]

[7] As an aside, let $\mathbf{i}\theta$ be the angle between a vector $\mathbf{a}$ and a blade $\mathbf{B}$. (See Fig. 2.) Then $\mathbf{i}\tan(\theta) = \mathbf{a}_\perp\mathbf{a}_\parallel^{-1}$ (opposite over adjacent). To see this, note that $\mathbf{a}_\perp\mathbf{a}_\parallel^{-1}$, the product of orthogonal vectors, is a bivector in the plane $\mathbf{i}$ they span. And $|\mathbf{a}_\perp\mathbf{a}_\parallel^{-1}| = |\mathbf{a}_\perp||\mathbf{a}_\parallel|^{-1} = \tan\theta$, where θ is the scalar angle between $\mathbf{a}$ and $\mathbf{B}$.

2.2.5. Pauli's electron theory. In 1927 Wolfgang Pauli published a quantum theory of an electron interacting with an electromagnetic field. Pauli's theory does not take Einstein's special relativity theory into account. A year later Paul Dirac published a relativistic quantum theory of the electron. We compare the VA and GA formulations of Pauli's theory in this section and of Dirac's theory in Sec. 2.4.5.

An electron has a property called *spin*, with a fixed value $\frac{1}{2}$. For our purposes think of the electron as spinning about an axis $\mathbf{s}$. The Pauli and Dirac theories describe the position and spin axis of spin-$\frac{1}{2}$ particles in an electromagnetic field. The field can change the position and the axis. We consider here only the case of a particle at rest in a uniform but time varying magnetic field. This case is important, for example, in the theory of nuclear magnetic resonance.

In the VA formulation of classical electromagnetism, a magnetic field is represented by a vector $\mathbf{b} \in \mathbb{R}^3$. In the $\mathbb{G}^3$ formulation, the field is represented by the bivector $\mathbf{B} = -\mathbf{b}^*$, in the plane orthogonal to $\mathbf{b}$. We saw in Sec. 2.2.3 that $\mathbf{B}$ and its scalar multiples specify rotations.

The basic physical fact is that from time t to $t + dt$, $\mathbf{s}$ rotates through the angle $\gamma \mathbf{B}(t)\, dt$, where γ is a constant. (Thus if $\mathbf{B}$ is constant in time, then $\mathbf{s}$ precesses with respect to the plane $\mathbf{B}$ with constant angular speed $\gamma|\mathbf{B}|$.) From Sec. 2.2.3, the rotation is represented by the unit quaternion $\mathrm{e}^{-\frac{1}{2}\gamma\mathbf{B}(t)dt}$.

In Pauli's theory the spin is not represented by $\mathbf{s}(t)$, but by the unit quaternion $\psi(t)$ which rotates $\mathbf{s}(0)$ to $\mathbf{s}(t)$. From Sec. 2.2.2 we have the composition

$$\psi(t+dt) = \mathrm{e}^{-\frac{1}{2}\gamma\mathbf{B}(t)dt}\,\psi(t) \approx \Big(1 - \frac{\gamma}{2}\,\mathbf{B}(t)\,dt\Big)\,\psi(t)\,.$$

This gives the GA version of *Pauli's equation* (for the spin): $\psi\,' = -\frac{\gamma}{2}\mathbf{B}\,\psi$.

The formulation of Pauli's theory just outlined uses GA, a general purpose mathematical language. The VA formulation, the one taught and used almost everywhere, requires one to learn specialized mathematics. It represents a spin as a unit complex vector $\left(\begin{smallmatrix} a+bi \\ c+di \end{smallmatrix}\right)$. These vectors are isomorphic to the unit quaternions, and so represent rotations. But there is no hint of this in elementary treatments of the VA theory.

The VA formulation of Pauli's theory uses the three complex *Pauli matrices*

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Real linear combinations of these matrices, together with the identity matrix, generate the *Pauli algebra*. In the theory the σ_i are associated, but not identified, with orthogonal directions in $\mathbb{R}^3$. But they should be so identified. For the Pauli algebra is isomorphic to $\mathbb{G}^3$, with $\sigma_j \leftrightarrow \mathbf{e}_j$!

The obvious-at-a-glance GA identities $\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3 = \mathbf{I}$, $\mathbf{e}_1\mathbf{e}_2 = \mathbf{I}\,\mathbf{e}_3$, $\mathbf{e}_1\mathbf{e}_2 = -\mathbf{e}_2\mathbf{e}_1$, and $\mathbf{e}_1\mathbf{e}_2 - \mathbf{e}_2\mathbf{e}_1 = 2\,\mathbf{I}\,\mathbf{e}_3$ correspond to important but not-so-obvious matrix identities of the σ_j. (Note that $\mathbf{I} = \mathbf{e}_1\mathbf{e}_2\mathbf{e}_3 \leftrightarrow \sigma_1\sigma_2\sigma_3 = i\mathsf{I}$, where I is the identity matrix.)

The VA formulation uses the "vector" of matrices $\boldsymbol{\sigma} \equiv (\sigma_1, \sigma_2, \sigma_3)$ to associate a *vector* $\mathbf{u} = u_1\,\mathbf{e}_1 + u_2\,\mathbf{e}_2 + u_3\,\mathbf{e}_3$ with the *matrix* $\boldsymbol{\sigma} \cdot \mathbf{u} \equiv u_1\,\sigma_1 + u_2\,\sigma_2 + u_3\,\sigma_3$. The identity $(\boldsymbol{\sigma}\cdot\mathbf{u})(\boldsymbol{\sigma}\cdot\mathbf{v}) = (\mathbf{u}\cdot\mathbf{v})\,\mathsf{I} + i\,\boldsymbol{\sigma}\cdot(\mathbf{u}\times\mathbf{v})$ plays an important role. It is a clumsy way to express a fundamental geometrical fact. For the identity is a matrix representation of our fundamental GA identity $\mathbf{u}\mathbf{v} = \mathbf{u}\cdot\mathbf{v} + \mathbf{u}\wedge\mathbf{v}$, Eq. (1.5).

2.3 Linear Algebra

Here is a sampling of GA ideas in linear algebra.

2.3.1. Linear independence. An inspection of the proof of Theorem 1.6.2 shows that vectors $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k$ are linearly independent if and only if $\mathbf{u}_1 \wedge \mathbf{u}_2 \wedge \cdots \wedge \mathbf{u}_k \neq 0$. Geometrically, this states that the vectors span a parallelepiped of nonzero k-volume.

2.3.2. Outermorphisms. We have obtained formulas for the rotation of a vector $\mathsf{R}_{\mathbf{i}\theta}(\mathbf{u})$ (Eq. (2.4)), the projection of a vector $\mathsf{P}_{\mathbf{B}}(\mathbf{u})$ (Eq. (1.14)), and the reflection of a vector $\mathsf{F}_{\mathbf{B}}(\mathbf{u})$ (Theorem (1.5.11a):

$$\begin{aligned} \mathsf{R}_{\mathbf{i}\theta}(\mathbf{u}) &= \mathrm{e}^{-\mathbf{i}\theta/2}\mathbf{u}\mathrm{e}^{\mathbf{i}\theta/2}, \\ \mathsf{P}_{\mathbf{B}}(\mathbf{u}) &= (\mathbf{u} \cdot \mathbf{B})/\mathbf{B}, \\ \mathsf{F}_{\mathbf{B}}(\mathbf{u}) &= (-1)^{k+1}\mathbf{B}\mathbf{u}\mathbf{B}^{-1} \ \ (k = \mathrm{grade}(\mathbf{B})). \end{aligned}$$

We can extend the formulas to rotate, project, and reflect higher dimensional objects. The figures below show that $\mathbf{u} \wedge \mathbf{v}$ rotated is $\mathsf{R}_{\mathbf{i}\theta}(\mathbf{u}) \wedge \mathsf{R}_{\mathbf{i}\theta}(\mathbf{v})$, $\mathbf{u} \wedge \mathbf{v}$ projected on the bivector $\mathbf{B}$ is $\mathsf{P}_{\mathbf{B}}(\mathbf{u}) \wedge \mathsf{P}_{\mathbf{B}}(\mathbf{v})$, $\mathbf{u} \wedge \mathbf{v}$ reflected in the bivector $\mathbf{B}$ is $\mathsf{F}_{\mathbf{B}}(\mathbf{u}) \wedge \mathsf{F}_{\mathbf{B}}(\mathbf{v})$.

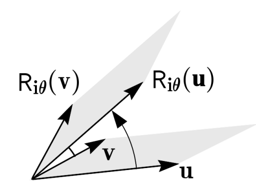


Fig. 4: $\mathsf{R}_{\mathbf{i}\theta}(\mathbf{u} \wedge \mathbf{v}) = \mathsf{R}_{\mathbf{i}\theta}(\mathbf{u}) \wedge \mathsf{R}_{\mathbf{i}\theta}(\mathbf{v})$.

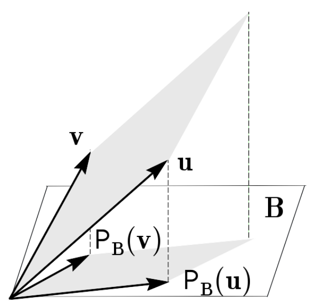


Fig. 5: $\mathsf{P}_{\mathbf{B}}(\mathbf{u} \wedge \mathbf{v}) = \mathsf{P}_{\mathbf{B}}(\mathbf{u}) \wedge \mathsf{P}_{\mathbf{B}}(\mathbf{v})$.

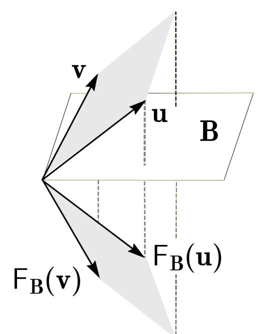


Fig. 6: $\mathsf{F}_{\mathbf{B}}(\mathbf{u} \wedge \mathbf{v}) = \mathsf{F}_{\mathbf{B}}(\mathbf{u}) \wedge \mathsf{F}_{\mathbf{B}}(\mathbf{v})$.

Thus we need to extend $\mathsf{R}_{\mathbf{i}\theta}$, $\mathsf{P}_{\mathbf{B}}$, $\mathsf{F}_{\mathbf{B}}$ according to the captions of the figures. A linear transformation f on $\mathbb{R}^n$ always extends uniquely to a linear transformation $\underline{\mathsf{f}}$ on $\mathbb{G}^n$ satisfying $\underline{\mathsf{f}}(A \wedge B) = \underline{\mathsf{f}}(A) \wedge \underline{\mathsf{f}}(B)$ for all multivectors A and B. The extension is called an *outermorphism*. The extensions satisfy $\underline{\mathsf{f}\,\mathsf{g}} = \underline{\mathsf{f}}\,\underline{\mathsf{g}}$ and $\underline{\mathsf{f}^{-1}} = \underline{\mathsf{f}}^{-1}$.

Henceforth we will drop the underbar and let f denote both a linear transformation and its outermorphism extension.

For vectors $\mathbf{u}$ and $\mathbf{v}$, $\mathsf{f}(\mathbf{u} \wedge \mathbf{v}) = \mathsf{f}(\mathbf{u}) \wedge \mathsf{f}(\mathbf{v})$ is a way to express algebraically the geometric fact that a linear transformation maps the parallelogram $\mathbf{u} \wedge \mathbf{v}$ to the parallelogram $\mathsf{f}(\mathbf{u}) \wedge \mathsf{f}(\mathbf{v})$.

There is no analog in VA, even in $\mathbb{R}^3$ when representing planes by normal vectors. For example, let f transform the plane $\mathbf{e}_1 \wedge \mathbf{e}_2$ to some other plane and $\mathbf{e}_3$ to itself. Then $\mathsf{f}(\mathbf{e}_1 \times \mathbf{e}_2) \neq \mathsf{f}(\mathbf{e}_1) \times \mathsf{f}(\mathbf{e}_2)$.

The outermorphism extensions of $\mathsf{R}_{\mathbf{i}\theta}$, $\mathsf{P}_{\mathbf{B}}$, $\mathsf{F}_{\mathbf{B}}$ to blades $\mathbf{A}$ are

$$\begin{aligned} \mathsf{R}_{\mathbf{i}\theta}(\mathbf{A}) &= \mathrm{e}^{-\mathbf{nI}\theta/2}\mathbf{A}\mathrm{e}^{\mathbf{nI}\theta/2}, \\ \mathsf{P}_{\mathbf{B}}(\mathbf{A}) &= (\mathbf{A} \cdot \mathbf{B})/\mathbf{B}, \\ \mathsf{F}_{\mathbf{B}}(\mathbf{A}) &= (-1)^{j(k+1)}\mathbf{B}\mathbf{A}\mathbf{B}^{-1} \quad (j = \mathrm{grade}(\mathbf{A}). \end{aligned}$$

Outermorphisms are grade preserving. For we have

$$\mathsf{f}(\mathbf{e}_1 \wedge \cdots \wedge \mathbf{e}_k) = \mathsf{f}(\mathbf{e}_1) \wedge \cdots \wedge \mathsf{f}(\mathbf{e}_k).$$

If $\mathsf{f}(\mathbf{e}_1), \ldots, \mathsf{f}(\mathbf{e}_k)$ are linearly independent, then from Theorem 1.6.2, the right side is a non-zero k-vector. If they are linearly dependent, then the right side is 0, again a k-vector.

2.3.3. Determinants. Since $\mathbf{I}$ is an n-vector and f is grade preserving, $\mathsf{f}(\mathbf{I})$ is also an n-vector. It is thus a multiple of $\mathbf{I}$: $\mathsf{f}(\mathbf{I}) = \det(\mathsf{f})\,\mathbf{I}$, where we have defined the *determinant* $\det(\mathsf{f})$ of f. This simple definition tells us what the determinant of a linear transformation *is*: the factor by which it multiplies n-volumes. Compare this to the usual algebraic definition of the determinant of the *matrix* of the transformation.

The GA definition makes obvious the product rule: $\det(\mathsf{f}\,\mathsf{g}) = \det(\mathsf{f})\det(\mathsf{g})$.

2.3.4. Theorem. Let $\{\mathbf{f}_1, \mathbf{f}_2, \ldots, \mathbf{f}_n\}$ be vectors in $\mathbb{R}^n$. Let $\mathcal{E} = \{\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n\}$ be an orthonormal basis for $\mathbb{R}^n$. Then

$$\mathbf{f}_1 \wedge \mathbf{f}_2 \wedge \cdots \wedge \mathbf{f}_n = \det \begin{bmatrix} \mathbf{f}_1 \cdot \mathbf{e}_1 & \mathbf{f}_1 \cdot \mathbf{e}_2 & \cdots & \mathbf{f}_1 \cdot \mathbf{e}_n \\ \mathbf{f}_2 \cdot \mathbf{e}_1 & \mathbf{f}_2 \cdot \mathbf{e}_2 & \cdots & \mathbf{f}_2 \cdot \mathbf{e}_n \\ \vdots & \vdots & \vdots & \vdots \\ \mathbf{f}_n \cdot \mathbf{e}_1 & \mathbf{f}_n \cdot \mathbf{e}_2 & \cdots & \mathbf{f}_n \cdot \mathbf{e}_n \end{bmatrix} \mathbf{I}. \tag{2.5}$$

To see this, define a linear transformation f by $\mathsf{f}(\mathbf{e}_i) = \mathbf{f}_i$. Then both sides of the equation are equal to $\det(\mathsf{f})\mathbf{I}$.

This is the key to proving standard results about determinants of matrices. For example, swapping two rows or two columns of a matrix changes the sign of its determinant.

2.3.5. Eigenblades. The linear transformation $\mathsf{f}(\mathbf{e}_1) = 2\,\mathbf{e}_2,\ \mathsf{f}(\mathbf{e}_2) = -3\,\mathbf{e}_1$ has no real eigenvalues. It does have complex eigenvalues, but they provide no geometric insight. In GA, f has *eigenblade* $\mathbf{e}_1 \wedge \mathbf{e}_2$ with eigenvalue 6: $\mathsf{f}(\mathbf{e}_1 \wedge \mathbf{e}_2) = 6(\mathbf{e}_1 \wedge \mathbf{e}_2)$. The transformation multiplies areas in the plane $\mathbf{e}_1 \wedge \mathbf{e}_2$ by 6.

2.3.6. Cramer's rule. Problem: In $\mathbb{R}^4$ solve $\mathbf{v} = c_1\mathbf{u}_1 + c_2\mathbf{u}_2 + c_3\mathbf{u}_3 + c_4\mathbf{u}_4$ for, say, c_2. Solution: Outer multiply the equation by $\mathbf{u}_1$ on the left and $\mathbf{u}_3 \wedge \mathbf{u}_4$ on the right. Then use $\mathbf{u}_i \wedge \mathbf{u}_i = 0$:

$$\mathbf{u}_1 \wedge \mathbf{v} \wedge \mathbf{u}_3 \wedge \mathbf{u}_4 = c_2\,(\mathbf{u}_1 \wedge \mathbf{u}_2 \wedge \mathbf{u}_3 \wedge \mathbf{u}_4)\,.$$

If the $\mathbf{u}_i$ are linearly independent, then the outer product on the right is invertible. Then we have a *symbolic* formula for c_2 which can be further manipulated.

2.3.7. Fourier expansion. Denote a member of a canonical basis by $\mathbf{e}_J$, where J is a set of increasing indexes. For example, if $J = \{2, 4\}$, then $\mathbf{e}_J = \mathbf{e}_2\,\mathbf{e}_4$. Since $\mathbf{e}_J^{-1} = \pm\mathbf{e}_J$, the $\mathbf{e}_J^{-1}$'s form a basis of $\mathbb{G}^n$. Multiply the expansion $A = \sum_J a_J\mathbf{e}_J^{-1}$ on the left by $\mathbf{e}_K$, take the scalar part, and use $\langle\, \mathbf{e}_K\mathbf{e}_J^{-1}\,\rangle_0 = \delta_K^J$ to give $\langle\, \mathbf{e}_K A\,\rangle_0 = a_K$. This gives the *Fourier expansion* $A = \sum_J \langle\, \mathbf{e}_J A\,\rangle_0\, \mathbf{e}_J^{-1}$.

2.4 The Spacetime Algebra

2.4.1. Indefinite metrics. GA readily extends from $\mathbb{R}^n$ to vector spaces with an indefinite metric. In $\mathbb{R}^{p,q}$ every orthonormal basis has p $\mathbf{e}_i$'s with $\mathbf{e}_i \cdot \mathbf{e}_i = 1$ and q $\mathbf{e}_i$'s with $\mathbf{e}_i \cdot \mathbf{e}_i = -1$. Correspondingly, in $\mathbb{G}^{p,q}$ there are p $\mathbf{e}_i$'s with $\mathbf{e}_i \cdot \mathbf{e}_i = \mathbf{e}_i^2 = 1$ and q $\mathbf{e}_i$'s with $\mathbf{e}_i \cdot \mathbf{e}_i = \mathbf{e}_i^2 = -1$. Many properties of $\mathbb{G}^n$ remain valid in $\mathbb{G}^{p,q}$. In particular, we still have Eq. (1.2): $\mathbf{vu} = -\mathbf{uv}$ for orthogonal vectors $\mathbf{u}$ and $\mathbf{v}$.

2.4.2. The spacetime algebra. Special relativity assigns *spacetime coordinates* (t, x, y, z) to a definite time and place. In VA, spacetime is represented by $\mathbb{R}^{1,3}$. An orthonormal basis has a time direction $\mathbf{e}_0$ with $\mathbf{e}_0 \cdot \mathbf{e}_0 = 1$ and three space directions $\mathbf{e}_i$ with $\mathbf{e}_i \cdot \mathbf{e}_i = -1$. In GA, spacetime is represented by the *spacetime algebra* $\mathbb{G}^{1,3}$.

To get very far in 4D relativistic physics, methods beyond VA must be employed, usually the tensor algebra or exterior algebra of $\mathbb{R}^{1,3}$, which are very different from anything a student has seen before in VA. The transition from $\mathbb{G}^3$ to $\mathbb{G}^{1,3}$ is easier.

2.4.3. Boosts. Consider a second coordinate system (t', x', y', z') whose spatial points (x', y', z') move with respect to those of the first with velocity $\mathbf{v}$. The origins $(0, 0, 0, 0)$ of the two systems coincide. And their spatial axes are parallel.

You can think of the $\{\mathbf{e}_i'\}$ basis as moving with the new coordinate system. However, translating a vector does not change it, and the $\mathbf{e}_i'$ are, with the $\mathbf{e}_i$, fixed vectors in $\mathbb{R}^{1,3}$.

The transformation $\mathbf{p} = \sum_{i=0}^{3} a_i \mathbf{e}_i \to \mathbf{p}' = \sum_{i=0}^{3} a_i \mathbf{e}_i'$ maps $\mathbf{p}$ to the vector $\mathbf{p}'$ which "looks the same" in the moving system. This active transformation is called a *boost*, or *Lorentz transformation*. Fig. 7 illustrates an analogy to a rotation in a plane.

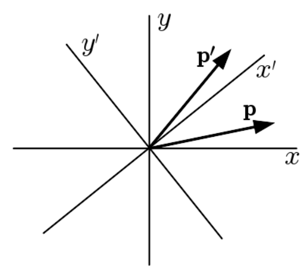


Fig. 7: An active "boost" $\mathbf{p} \to \mathbf{p}'$ in a plane.

The analogy is appropriate. For it can be shown that the boost is a rotation in the $\mathbf{e}_0\mathbf{v}$ plane: $\mathbf{p} \to e^{-\mathbf{e}_0\hat{\mathbf{v}}\alpha/2}\,\mathbf{p}\,e^{\mathbf{e}_0\hat{\mathbf{v}}\alpha/2}$. (C. f., Eq. (2.4).) Here we have set $\mathbf{v} = \tanh(\alpha)\hat{\mathbf{v}}$, where $\hat{\mathbf{v}}^2 = -1$.[8] And the exponential function is defined for all multivectors M by $e^M = \sum_{i=0}^{\infty} M^i/i!$. You can verify that $e^{\mathbf{e}_0\hat{\mathbf{v}}\alpha/2} = \cosh(\alpha/2) + \mathbf{e}_0\hat{\mathbf{v}}\sinh(\alpha/2)$.

VA represents boosts with 4×4 orthogonal matrices on $\mathbb{R}^{1,3}$. The GA exponential function representation of boosts in $\mathbb{G}^{1,3}$ has advantages similar to the GA exponential function representation of rotations in $\mathbb{R}^3$. (See Sec. 2.2.2.) We will see an example in the next section, where we compute the composition of two boosts. The results obtained there in a few lines are much more difficult to obtain with VA, so difficult that it is a rare relativity text which derives them.

[8] We use units of space and time in which the speed of light $c = 1$. For example, time might be measured in seconds and distance in light-seconds. Since speeds of material objects are less than c, $-1 < \mathbf{v}^2 \leq 0$. Thus $\tanh(\alpha) = (-\mathbf{v}^2)^{\frac{1}{2}}$.

2.4.4. Composition of boosts. By definition, a boost is free of any spatial rotation. Perhaps surprisingly, a composition of boosts is not. The composition of boosts $\mathrm{e}^{-\mathbf{e}_0\hat{\mathbf{v}}\beta/2}$ and $\mathrm{e}^{-\mathbf{e}_0\hat{\mathbf{u}}\alpha/2}$ can be written as a spatial rotation followed by a boost:

$$\mathrm{e}^{-\mathbf{e}_0\hat{\mathbf{v}}\beta/2}\,\mathrm{e}^{-\mathbf{e}_0\hat{\mathbf{u}}\alpha/2} = \mathrm{e}^{-\mathbf{e}_0\hat{\mathbf{w}}\delta/2}\,\mathrm{e}^{-\mathbf{i}\theta/2}. \tag{2.6}$$

Active transformations do not change reference frames, so we can express the boosts in Eq. (2.6) using the common basis vector $\mathbf{e}_0$ and vectors orthogonal to it ($\hat{\mathbf{u}}, \hat{\mathbf{v}}, \hat{\mathbf{w}}$).

The rotation $\mathrm{e}^{-\mathbf{i}\theta/2}$ is called a *Thomas rotation*. It is an important physical effect.

We prove Eq. (2.6) by solving it uniquely for $\hat{\mathbf{w}}, \delta, \mathbf{i}, \theta$ on the right side. Expand the exponentials and equate the terms with an $\mathbf{e}_0$ component and those without:

$$\begin{aligned} s_\alpha c_\beta \hat{\mathbf{u}} + c_\alpha s_\beta \hat{\mathbf{v}} &= s_\delta \hat{\mathbf{w}}\,\mathrm{e}^{-\mathbf{i}\theta/2}, \\ c_\alpha c_\beta - s_\alpha s_\beta \hat{\mathbf{v}}\hat{\mathbf{u}} &= c_\delta\,\mathrm{e}^{-\mathbf{i}\theta/2}, \end{aligned} \tag{2.7}$$

where $s_\alpha = \sinh(\alpha/2),\ c_\beta = \cosh(\beta/2)$, etc. Divide to obtain $\hat{\mathbf{w}}$ and δ:[9]

$$\tanh(\delta/2)\,\hat{\mathbf{w}} = \frac{s_\alpha c_\beta \hat{\mathbf{u}} + c_\alpha s_\beta \hat{\mathbf{v}}}{c_\alpha c_\beta - s_\alpha s_\beta \hat{\mathbf{v}}\hat{\mathbf{u}}} = \frac{\tanh(\alpha/2)\hat{\mathbf{u}} + \tanh(\beta/2)\hat{\mathbf{v}}}{1 - \tanh(\alpha/2)\tanh(\beta/2)\hat{\mathbf{v}}\hat{\mathbf{u}}}\,. \tag{2.8}$$

When the boosts are parallel, this reduces to the familiar "addition of velocities" formula in special relativity.

Equate the bivector parts and the scalar parts of Eq. (2.7):

$$\begin{aligned} s_\alpha s_\beta\, \hat{\mathbf{v}} \wedge \hat{\mathbf{u}} &= c_\delta \sin(\theta/2)\,\mathbf{i}, \\ c_\alpha c_\beta - s_\alpha s_\beta\, \hat{\mathbf{v}} \cdot \hat{\mathbf{u}} &= c_\delta \cos(\theta/2). \end{aligned}$$

Divide to obtain $\mathbf{i}$ and θ:

$$\tan(\theta/2)\,\mathbf{i} = \frac{s_\alpha s_\beta\, \hat{\mathbf{v}} \wedge \hat{\mathbf{u}}}{c_\alpha c_\beta - s_\alpha s_\beta\, \hat{\mathbf{v}} \cdot \hat{\mathbf{u}}} = \frac{\tanh(\alpha/2)\tanh(\beta/2)\, \hat{\mathbf{v}} \wedge \hat{\mathbf{u}}}{1 - \tanh(\alpha/2)\tanh(\beta/2)\, \hat{\mathbf{v}} \cdot \hat{\mathbf{u}}}\,.$$

The rotation plane $\mathbf{i}$ is $\hat{\mathbf{v}} \wedge \hat{\mathbf{u}}$. To obtain a scalar expression for $\tan(\theta/2)$, substitute $\hat{\mathbf{v}} \wedge \hat{\mathbf{u}} = \sin\phi\mathbf{i}$ and $\hat{\mathbf{v}} \cdot \hat{\mathbf{u}} = -\cos\phi$, where ϕ is the scalar angle from $\hat{\mathbf{v}}$ to $\hat{\mathbf{u}}$.

2.4.5. Dirac's electron theory. The most elegant formulation of Dirac's relativistic quantum theory of the electron is in the spacetime algebra.

Recall from Sec. 2.2.5 that Pauli's theory represents spins by 3D rotations, i.e., by members of $\mathsf{SO}(3)$. Dirac's theory represents spins by members of $\mathsf{SO}(1,3)$.

The VA version of Dirac's equation uses the four 4×4 complex *Dirac matrices* $\gamma_0, \gamma_1, \gamma_2, \gamma_3$ associated with (but not identified with) orthogonal directions in spacetime. These matrices generate the *Dirac algebra*. The Dirac algebra is isomorphic to $\mathbb{G}^{1,3}$, with $\gamma_j \leftrightarrow \mathbf{e}_j$.

Both Pauli and Dirac invented the geometric algebra for the space(time) in which they were working out of necessity, without realizing that the algebras are not special to quantum theory, but have deep geometric significance and wide applicability. From the perspective of GA, the Pauli and Dirac matrix algebras are uninteresting representations of a geometric algebra, which obscure the physical content of their theories.

[9] To see that the right side of Eq. (2.8) is in fact a vector, multiply its numerator and denominator by $1 - \tanh(\alpha/2)\tanh(\beta/2)\hat{\mathbf{u}}\hat{\mathbf{v}}$. Then use Theorem 1.5.11a and $\hat{\mathbf{u}} \cdot \hat{\mathbf{v}} = \frac{1}{2}(\hat{\mathbf{u}}\hat{\mathbf{v}} + \hat{\mathbf{v}}\hat{\mathbf{u}})$.

3 Geometric Calculus

3.1 Derivatives

3.1.1. The norm. Expand a multivector A with respect to a canonical basis: $A = \sum_J a_J \mathbf{e}_J$. Then the *norm* of A is defined by $|A|^2 = \sum_J |a_J|^2$. Limits in geometric calculus are with respect to this norm.

Of course $|A + B| \leq |A| + |B|$. It is easy to see that if $\mathbf{A} \subseteq \mathbf{B}$ then $|\mathbf{AB}| = |\mathbf{A}|\,|\mathbf{B}|$.

3.1.2. The gradient. Define the *gradient* $\boldsymbol{\nabla} = \sum_j \mathbf{e}_j \partial_j$, where ∂_j is a partial derivative. The gradient acts algebraically as a vector: we can multiply it by a scalar field f, giving the vector field $\boldsymbol{\nabla} f$; dot it with a vector field $\mathbf{f}$, giving the scalar field $\boldsymbol{\nabla} \cdot \mathbf{f}$; and cross it with $\mathbf{f}$, giving the vector field $\boldsymbol{\nabla} \times \mathbf{f}$. But $\boldsymbol{\nabla}\mathbf{f}$, a product of vectors, cannot be formed in vector calculus.

In geometric calculus $\boldsymbol{\nabla}\mathbf{f}$ *does* make sense. This product of vectors is scalar + bivector: $\boldsymbol{\nabla}\mathbf{f} = \boldsymbol{\nabla} \cdot \mathbf{f} + \boldsymbol{\nabla} \wedge \mathbf{f}$, just as with our fundamental identity, Eq. (1.5). In this way $\boldsymbol{\nabla}$ unifies the divergence and curl and generalizes the curl to nD.

The geometric calculus identity $\boldsymbol{\nabla}^2 \mathbf{f} = \boldsymbol{\nabla}(\boldsymbol{\nabla}\mathbf{f})$ cannot be written in vector calculus. Instead, we must resort to $\boldsymbol{\nabla}^2 \mathbf{f} = \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{f}) - \boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{f})$ – and this only in $\mathbb{R}^3$.

3.1.3. Analytic functions. Let $f(x, y) = u(x, y) + v(x, y)\,\mathbf{i}$, with u and v real valued. Then

$$\boldsymbol{\nabla} f = \mathbf{e}_1(u_x + v_x\mathbf{i}) + \mathbf{e}_2(u_y + v_y\mathbf{i}) = \mathbf{e}_1(u_x - v_y) + \mathbf{e}_2(v_x + u_y)\,.$$

From the Cauchy-Riemann equations, $\boldsymbol{\nabla} f = 0 \Leftrightarrow f$ is analytic. Generalizing, we call a multivector field F on $\mathbb{R}^n$ *analytic* if $\boldsymbol{\nabla} F = 0$.

This definition leads to a generalization of standard complex analysis to n (real) dimensions. Many standard results generalize. A simple example: since $\boldsymbol{\nabla} F = 0 \;\Rightarrow\; \boldsymbol{\nabla}^2 F = \boldsymbol{\nabla}(\boldsymbol{\nabla}(F)) = 0$, analytic functions are harmonic functions. Most important, Cauchy's theorem and Cauchy's integral formula generalize, as we shall see.

3.1.4. Generalize $\boldsymbol{\nabla}$. It violates the spirit of GA to write f above as a function of the coordinates (x, y). Henceforth we shall think of it, equivalently, as a function of the vector $\mathbf{x} = \mathbf{e}_1 x + \mathbf{e}_2 y$, and similarly in higher dimensions. As an example, you can verify that $\boldsymbol{\nabla}(\mathbf{x}\,\mathbf{a}) = n\,\mathbf{a}$.

With this change of viewpoint we can generalize $\boldsymbol{\nabla}$. First, the *directional derivative* of F in the "direction" A is

$$\partial_A F(X) = \lim_{\tau \to 0} \frac{F(X + \tau A) - F(X)}{\tau}\,.$$

If A contains grades for which F is not defined, then define $\partial_A F(X) = 0$. For example, if F is a function of a vector $\mathbf{x}$, then $\partial_{\mathbf{e}_1\mathbf{e}_2} F(\mathbf{x}) = 0$.

We can now generalize the gradient $\boldsymbol{\nabla} = \sum_j \mathbf{e}_j\, \partial_{\mathbf{e}_j}$ above:

$$\boldsymbol{\nabla} = \sum_J \mathbf{e}_J^{-1}\, \partial_{\mathbf{e}_J}\,.$$

This is a Fourier expansion of $\boldsymbol{\nabla}$. (See Sec. 2.3.7.) The derivative $\boldsymbol{\nabla}$ is independent of the basis $\{\mathbf{e}_i\}$.

3.1.5. Minimization example. As an example of the use of the generalized $\boldsymbol{\nabla}$, consider the problem of rotating 3D vectors $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n$ to best approximate the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ [7]. By this we mean choose a unit quaternion R to minimize $\phi(R) = \sum_j (\mathbf{v}_j - R\mathbf{u}_j R^{-1})^2$. We show that the minimum occurs when $\boldsymbol{\nabla}\phi(R) = 0$.

First note that we may drop the constraint $|R| = 1$, since $\phi(aR) = \phi(R)$. Set $R = r_0 + \mathbf{e}_3\mathbf{e}_2 r_1 + \mathbf{e}_1\mathbf{e}_3 r_2 + \mathbf{e}_2\mathbf{e}_1 r_3$. Then $\phi(R) = \phi(r_0 + \mathbf{e}_3\mathbf{e}_2 r_1 + \mathbf{e}_1\mathbf{e}_3 r_2 + \mathbf{e}_2\mathbf{e}_1 r_3)$ is a real valued function of four real variables. Its minimum occurs when all four partial derivatives $\partial\phi/\partial r_i = 0$. We have

$$\boldsymbol{\nabla}\phi(R) = 1\,\partial\phi/\partial r_0 + \mathbf{e}_2\mathbf{e}_3\,\partial\phi/\partial r_1 + \mathbf{e}_3\mathbf{e}_1\,\partial\phi/\partial r_2 + \mathbf{e}_1\mathbf{e}_2\,\partial\phi/\partial r_3.$$

Thus the minimum occurs when $\boldsymbol{\nabla}\phi(R) = 0$.

After learning a few rules about differentiating with respect to a multivector we find that $\boldsymbol{\nabla}\phi(R) = 0 \;\Rightarrow\; \sum_j \mathbf{v}_j \wedge (R\mathbf{u}_j R^{-1}) = 0$. The right side has an agreeable geometric interpretation: the (bivector) sum of the parallelograms spanned by the $\mathbf{v}_j$ and the rotated $\mathbf{u}_j$ is zero. The equation can be solved for R.

Geometric calculus uses only one derivative, $\boldsymbol{\nabla}\phi$, to solve this problem. Vector calculus must break $\boldsymbol{\nabla}\phi$ into its four components $\partial\phi/\partial r_i$. Is generally best not to break a multivector into its components, just as it is generally best not to break a complex number into its real and imaginary parts or a vector into its components.

3.1.6. Electromagnetism. Elementary electromagnetic theory is usually formulated in 3D vector calculus. Two vector fields, the electric field $\mathbf{e}$ and the magnetic field $\mathbf{b}$, represent the electromagnetic field. The charge density scalar field ρ and the current density vector field $\mathbf{j}$ represent the distribution and motion of charges. *Maxwell's equations* are the heart of the theory:

$$\boldsymbol{\nabla}\cdot\mathbf{e} = 4\pi\rho, \quad \boldsymbol{\nabla}\times\mathbf{e} = -\frac{\partial\mathbf{b}}{\partial t}, \quad \boldsymbol{\nabla}\cdot\mathbf{b} = 0, \quad \boldsymbol{\nabla}\times\mathbf{b} = 4\pi\mathbf{j} + \frac{\partial\mathbf{e}}{\partial t}. \tag{3.1}$$

The spacetime algebra $\mathbb{G}^{1,3}$ of Sec. 2.4.2 provides a more elegant formulation. A spacetime bivector field F unifying $\mathbf{e}$ and $\mathbf{b}$ represents the electromagnetic field. A spacetime vector field J unifying ρ and $\mathbf{j}$ represents the distribution and motion of charges. Maxwell's four equations become a single equation: $\boldsymbol{\nabla}F = J$. What a simple equation: the derivative of one single grade field is another.

Multiplying $\boldsymbol{\nabla}F = J$ by $\mathbf{e}_0$ and equating the 0-, 1-, 2-, and 3-vector parts yields the standard Maxwell equations, Eqs. (3.1).

Calculations using the $\mathbb{G}^{1,3}$ formulation of Maxwell's equations are often easier, and sometimes much easier, than using the $\mathbb{R}^3$ formulation. This is in part due to the fact that the GA derivative $\boldsymbol{\nabla}$, unlike the divergence and curl in Eqs. (3.1), is invertible, as we will see in Sec. 3.2.4.

In geometric calculus the same derivative $\boldsymbol{\nabla}$ is used in the definition of an analytic function, the minimization example, Maxwell's theory, the full Pauli theory, and the full Dirac theory. That's unification.

3.2 Integrals

Let M be a compact oriented m-dimensional manifold with boundary in $\mathbb{R}^n$.

3.2.1. The directed integral. Let $F(\mathbf{x})$ be a multivector valued field on M. Then we can form the *directed integral*

$$\int_M d^m\mathbf{x}\, F. \tag{3.2}$$

Here $d^m\mathbf{x} = \mathbf{I}_m(\mathbf{x})d^m x$, where $d^m x$ is an element of m-volume of M at $\mathbf{x}$ and $\mathbf{I}_m(\mathbf{x})$ is the pseudoscalar of the tangent space to M at $\mathbf{x}$. For example, if M is a surface in $\mathbb{R}^3$, then $d^2x = dS$ is an element of area of M and $\mathbf{I}_2(\mathbf{x})$ is the pseudoscalar of the tangent plane to M at $\mathbf{x}$ (a bivector). If M is a volume in $\mathbb{R}^3$, then $d^3x = dV$ is an element of volume of V and $\mathbf{I}_3(\mathbf{x}) \equiv \mathbf{I}_3$ is the pseudoscalar of $\mathbb{R}^3$. Note that the order of the factors in the integrand is important, as the geometric product is not commutative.

The integral $\int_C f(z)\, dz$ from complex analysis is a directed integral, a special case of Eq. (3.2).

3.2.2. The fundamental theorem. The *vector derivative* $\boldsymbol{\partial}$ can be thought of as the projection of $\boldsymbol{\nabla}$ on M. Let F be a multivector valued function defined on M.

The Fundamental Theorem of (Geometric) Calculus

$$\int_M d^m\mathbf{x}\, \boldsymbol{\partial} F = \int_{\partial M} d^{m-1}\mathbf{x}\, F. \tag{3.3}$$

This is a marvelous theorem. Its scalar part is equivalent to Stokes' theorem for differential forms. Thus the divergence and Stokes' theorems of vector calculus are special cases of Eq. (3.3). A generalization of Cauchy's theorem to a manifold is an obvious special case: If F is analytic on M, i.e., if $\boldsymbol{\partial} F = 0$, then $\int_{\partial M} d^{m-1}\mathbf{x}\, F = 0$.

The fundamental theorem also generalizes the residue theorem. Let Ω_n be the $(n-1)$-volume of the boundary of the unit ball in $\mathbb{R}^n$ (e.g., $\Omega_2 = 2\pi$). Let δ be Dirac's delta function. Then F has a *pole* at $\mathbf{x}_k$ with *residue* the multivector R_k if $\boldsymbol{\partial} F(\mathbf{x}) = \Omega_n R_k \delta(\mathbf{x} - \mathbf{x}_k)$ near $\mathbf{x}_k$. Eq. (3.3) holds if F is analytic in M except at a finite number of poles at the points $\mathbf{x}_k$. Thus

$$\int_{\partial M} d^{m-1}\mathbf{x}\, F = \int_M d^m\mathbf{x}\, \boldsymbol{\partial} F = \int_M d^m x\, \mathbf{I}(\mathbf{x}) \Big(\sum_k \Omega_n R_k \delta(\mathbf{x} - \mathbf{x}_k) \Big) = \Omega_n \sum_k \mathbf{I}(\mathbf{x}_k) R_k.$$

If M is a region of a plane and the R_k are complex numbers, then this reduces to the usual residue theorem.

With directed integrals on manifolds, complex analysis extends to manifolds of any dimension, and is a subdiscipline of real analysis: it is the study of functions F with $\boldsymbol{\partial} F = 0$. Traditional real analysis does not use directed integrals. Unification with complex analysis cannot be achieved without them. For example, consider Cauchy's theorem: $\oint_C f(z)\, dz = 0$. Green's theorem gives the real and imaginary parts of the theorem separately. But the theorem cannot be written as a single formula in vector calculus or with differential forms.

In $\mathbb{R}^n$ the fundamental theorem can be generalized, with several important corollaries. Eqs. (3.4) and (3.5) below are examples.

3.2.3. F from $\boldsymbol{\nabla} F$ and boundary values. Let F be a multivector valued function defined in a region V of $\mathbb{R}^n$. Then for $\mathbf{x}_0 \in V$,

$$F(\mathbf{x}_0) = \frac{(-1)^n}{\Omega_n \mathbf{I}} \left\{ \int_V \frac{\mathbf{x}-\mathbf{x}_0}{|\mathbf{x}-\mathbf{x}_0|^n} \, d^n\mathbf{x} \, \boldsymbol{\nabla} F(\mathbf{x}) \; - \int_{\partial V} \frac{\mathbf{x}-\mathbf{x}_0}{|\mathbf{x}-\mathbf{x}_0|^n} \, d^{n-1}\mathbf{x} \, F(\mathbf{x}) \right\}. \tag{3.4}$$

In particular, $F|_V$ is determined by $\boldsymbol{\nabla} F|_V$ and $F|_{\partial V}$. This is a generalization of Pompeiu's (not very well known) theorem of complex analysis.

If F is analytic, then Eq. (3.4) becomes

$$F(\mathbf{x}_0) = -\frac{(-1)^n}{\Omega_n \mathbf{I}} \int_{\partial V} \frac{\mathbf{x}-\mathbf{x}_0}{|\mathbf{x}-\mathbf{x}_0|^n} \, d^{n-1}\mathbf{x} \, F(\mathbf{x}) \, .$$

This is a generalization of Cauchy's integral formula.

3.2.4. Antiderivatives. Eq. (3.4) can be used to prove that F has an *antiderivative* $P\colon \boldsymbol{\nabla} P = F$. In other words, $\boldsymbol{\nabla}$ is invertible. Clearly, two antiderivatives of F differ by an analytic function. This generalizes the $n = 1$ case, where the analytic functions are the constant functions.

For $n > 2$ an antiderivative is given by

$$P(\mathbf{x}_0) = \frac{1}{(n-2)\,\Omega_n} \left\{ \int_V \frac{d^m x}{|\mathbf{x}-\mathbf{x}_0|^{n-2}} \, \boldsymbol{\nabla} F(\mathbf{x}) \; - \int_{\partial V} \frac{d^{m-1} x}{|\mathbf{x}-\mathbf{x}_0|^{n-2}} \, \mathbf{n}(\mathbf{x}) \, F(\mathbf{x}) \right\}, \tag{3.5}$$

where $\mathbf{n}(\mathbf{x}) = \pm \mathbf{I}(\mathbf{x})^*$ is the unit outward normal to ∂V. (In fact, $\mathbf{n} = \mathbf{I}^{-1}\mathbf{I}(\mathbf{x})$, where $\mathbf{I}$ is the pseudoscalar of $\mathbb{R}^n$.)

If F is analytic, then the first integral is zero.

If $|\,\boldsymbol{\nabla} F(\mathbf{x})\,| = O(\,|\,\mathbf{x}\,|^{-2})$ and $\lim_{\mathbf{x}\to\infty} \mathbf{x}\, F(\mathbf{x}) = 0$, then

$$P(\mathbf{x}_0) = \frac{1}{(n-2)\,\Omega_n} \int_{\mathbb{R}^n} \frac{d^m x}{|\mathbf{x}-\mathbf{x}_0|^{n-2}} \, \boldsymbol{\nabla} F(\mathbf{x}) \, .$$

3.2.5. Potentials, Fields, Sources. Three fields P, F, and S with $\boldsymbol{\nabla} P = F$ and $\boldsymbol{\nabla} F = S$ are called a *potential*, a *field*, and a *source*, respectively. Given a source S with suitable boundedness, P and F always exist. A common situation in $\mathbb{R}^3$ is $S = s - \mathbf{s}^*$, where s is a scalar field and $\mathbf{s}$ is a vector field with $\boldsymbol{\nabla} \cdot \mathbf{s} = 0$. Then we can take $P = p - \mathbf{p}^*$, where p is a scalar field, $\mathbf{p}$ is a vector field with $\boldsymbol{\nabla} \cdot \mathbf{p} = 0$, and $F = \mathbf{f}$, a vector field. Vector calculus cannot form S or P. In particular, it cannot use the simple formulas $F = \boldsymbol{\nabla} P$ and $S = \boldsymbol{\nabla} F$. Instead, $\mathbf{f} = \boldsymbol{\nabla} p - \boldsymbol{\nabla} \times \mathbf{p}$, $s = \boldsymbol{\nabla} \cdot \mathbf{f}$, and $\mathbf{s} = \boldsymbol{\nabla} \times \mathbf{f}$.

4 Other Models

4.1 The Homogeneous Model

We have represented subspaces of $\mathbb{R}^n$ (points, lines, planes, ... through the origin) with blades of $\mathbb{G}^n$. This is the *vector space model.* But we have not represented *translated* subspaces (points, lines, planes, ... not through the origin). Yet there is nothing special geometrically about the origin.

4.1.1. The homogeneous model. The homogeneous model represents and manipulates all points, lines, planes, ... much as the vector space model represents and manipulates those through the origin. It is the coordinate-free geometric algebra version of homogeneous coordinates used in computer graphics and projective geometry.

> The homogeneous model represents a translated k-dimensional subspace of $\mathbb{R}^n$, and k-volumes in it, with a $(k+1)$–blade of $\mathbb{G}^{n+1}$.

In particular, a point, which is a translation of the 0-dimensional subspace $\{\mathbf{0}\}$, is represented by a 1-blade, i.e., a vector. To see how this works, extend $\mathbb{R}^n$ with a unit vector e orthogonal to $\mathbb{R}^n$. Then we have $\mathbb{R}^{n+1}$, and with it $\mathbb{G}^{n+1}$.

> The homogeneous model represents a *point* P at the end of the vector $\mathbf{p} \in \mathbb{R}^n$ with the *vector* $p = e + \mathbf{p} \in \mathbb{R}^{n+1}$. (Only vectors in $\mathbb{R}^n$ will be denoted in bold.) Fig. 8 shows a useful way to visualize this for $\mathbb{R}^2$.

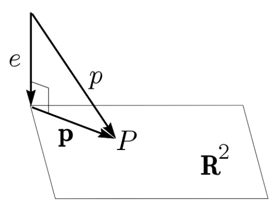


Fig. 8: Vector $p = e + \mathbf{p}$ represents the point P.

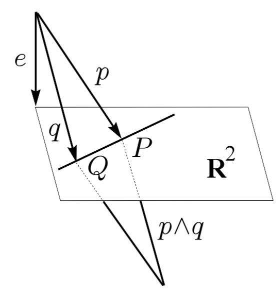
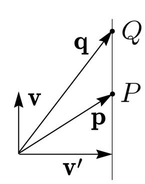


Fig. 9: Bivector $p \wedge q$ represents the oriented segment PQ.

The vector p is *normalized* in the sense that $p \cdot e = e \cdot e + \mathbf{p} \cdot e = 1$. However, the representation is *homogeneous*: for scalars $\lambda \neq 0$, $\lambda(e + \mathbf{p})$ also represents P.

> The homogeneous model represents the oriented *length* PQ with the *bivector* $p \wedge q$. See Fig. 9.

Let $\mathbf{v} = \mathbf{q} - \mathbf{p}$ and $\mathbf{v}'$ be the vector with endpoint on the line $\overline{PQ}$ and perpendicular to it. See the figure at right. Then $p \wedge q$ determines, and is determined by, $\mathbf{v}$ and $\mathbf{v}'$. This follows from $\mathbf{v} = e \cdot (p \wedge q)$ and $p \wedge q = (e + \mathbf{p}) \wedge \mathbf{v} = (e + \mathbf{v}') \wedge \mathbf{v} = (e + \mathbf{v}')\, \mathbf{v}$. The equation $p \wedge q = (e + \mathbf{v}') \wedge \mathbf{v}$ shows that $p \wedge q$ does not determine $\mathbf{p}$ or $\mathbf{q}$.



> The homogeneous model represents the oriented *area* PQR with the *trivector* $p \wedge q \wedge r$.

Let $\mathbf{V} = (\mathbf{q} - \mathbf{p}) \wedge (\mathbf{r} - \mathbf{p})$, which represents twice the area. Define $\mathbf{v}'$ as above. Then $p \wedge q \wedge r$ determines, and is determined by, $\mathbf{V}$ and $\mathbf{v}'$. The equation $p \wedge q \wedge r = (e + \mathbf{v}') \wedge \mathbf{V}$ shows that $p \wedge q \wedge r$ does not determine $\mathbf{p}$, $\mathbf{q}$, or $\mathbf{r}$.

In VA and the vector space model of GA, vectors do double duty, representing oriented line segments and points. (To represent points, place the tails of all vectors at some fixed origin. Then the heads of the vectors are in one-to-one correspondence with points.) In the homogeneous model oriented line segments and points have different representations.

We now have two geometric algebras for Euclidean n-space: the vector space model $\mathbb{G}^n$ and the homogeneous model $\mathbb{G}^{n+1}$. In both, blades represent geometric objects. In the vector space model a vector represents an oriented line segment. In the homogeneous model it represents a point. In the vector space model an outer product of vectors represents an oriented area. In the homogeneous model it represents an oriented length. Oriented areas and oriented lengths are different, yet they share a common algebraic structure. We have to learn geometric algebra only once to work with both.

4.1.2. The Euclidean group. Rotations and reflections generate the orthogonal group $\mathsf{O}(n)$. Include translations to generate the distance preserving Euclidean group. In $\mathbb{R}^n$ translations are not linear. This is a problem: "The orthogonal group is multiplicative while the translation group is additive, so combining the two destroys the simplicity of both."[4]

The homogeneous model solves the problem. The subgroup of $\mathsf{O}(n+1)$ which fixes e is isomorphic to $\mathsf{O}(n)$. For members of this subgroup map $\mathbb{R}^n$, the subspace of $\mathbb{R}^{n+1}$ orthogonal to e, to $\mathbb{R}^n$. And since $p \cdot q = 1 + \mathbf{p} \cdot \mathbf{q}$, the map is also orthogonal on $\mathbb{R}^n$.

For a fixed $\mathbf{a} \in \mathbb{R}^n$, consider the linear transformation $x \to x + (x \cdot e)\mathbf{a}$ of $\mathbb{R}^{n+1}$. In particular, $e + \mathbf{p} \to e + (\mathbf{p} + \mathbf{a})$. This represents a translation by $\mathbf{a}$ in $\mathbb{R}^n$. In this way translations are *linearized* in the homogeneous model.

4.1.3. Join and meet. Geometric objects *join* to form higher dimensional objects. For example, the join of two points is the line between them; the join of intersecting lines is the plane containing them; and the join of a line and a point not on it is the plane containing them.

Geometric objects *meet* in lower dimensional objects, their intersection. Thus the meet of an intersecting line and plane is their point or line of intersection, and the meet of two intersecting planes is their line of intersection.

GA defines the join and meet of two blades (only) to represent the join and meet of the geometric objects that they represent.

The *join* of blades $\mathbf{A}$ and $\mathbf{B}$ is the span of their subspaces. From Eq. (1.11), if $\mathbf{A} \cap \mathbf{B} = \{0\}$, then their join $\mathbf{J} = \mathbf{A} \wedge \mathbf{B}$. However, there is no general formula for the join in terms of the geometric product, as there is for the inner and outer products (Eqs. (1.7) and (1.9)).

The *meet* of $\mathbf{A}$ and $\mathbf{B}$ is the intersection of their subspaces. In this section, let $\mathbf{X}^* = \mathbf{X}/\mathbf{J}$, the dual of $\mathbf{X}$ in join $\mathbf{J}$ of $\mathbf{A}$ and $\mathbf{B}$. The meet of $\mathbf{A}$ and $\mathbf{B}$ is given by

$$\mathbf{A} \vee \mathbf{B} = \mathbf{A}^* \cdot \mathbf{B}\,. \tag{4.1}$$

Before proving this we give examples of the join and meet.

In the vector space model, two lines through the origin join to form a plane. Two examples are $\mathbf{e}_1 \wedge \mathbf{e}_2 = \mathbf{e}_1\mathbf{e}_2$ and $\mathbf{e}_2 \wedge \mathbf{e}_3 = \mathbf{e}_2\mathbf{e}_3$. The two planes meet in a line:

$$(\mathbf{e}_1\mathbf{e}_2) \vee (\mathbf{e}_2\mathbf{e}_3) = (\mathbf{e}_1\mathbf{e}_2)^* \cdot (\mathbf{e}_2\,\mathbf{e}_3) = -\mathbf{e}_2\,.$$

In the homogeneous model, two points join to form a line. Two examples are $\ell_1 = (e + \mathbf{e}_1) \wedge (e + \mathbf{e}_2)$ and $\ell_2 = e \wedge (e + \mathbf{e}_1 + \mathbf{e}_2)$. The two lines meet in a point:

$$\ell_1 \vee \ell_2 = (e\,\mathbf{e}_2 - e\,\mathbf{e}_1 + \mathbf{e}_1\mathbf{e}_2)^* \cdot (e\,\mathbf{e}_1 + e\,\mathbf{e}_2) = 2\left(e + \frac{1}{2}(\mathbf{e}_1 + \mathbf{e}_2)\right).$$

We now prove Eq. (4.1), $\mathbf{A} \vee \mathbf{B} = \mathbf{A}^* \cdot \mathbf{B}$. We work with the two sides of the equation separately, using duality, Eq. (1.12).

$$\begin{aligned} &\mathbf{x} \in \mathbf{A} \vee \mathbf{B} \Leftrightarrow (\mathbf{x} \in \mathbf{A} \text{ and } \mathbf{x} \in \mathbf{B}) \Leftrightarrow (\mathbf{x} \perp \mathbf{A}^* \text{ and } \mathbf{x} \perp \mathbf{B}^*), \\ &\mathbf{x} \in \mathbf{A}^* \cdot \mathbf{B} \Leftrightarrow \mathbf{x} \in (\mathbf{A}^* \wedge \mathbf{B}^*)^* \Leftrightarrow \mathbf{x} \perp \mathbf{A}^* \wedge \mathbf{B}^*. \end{aligned} \tag{4.2}$$

To complete the proof we show that the right sides above are equivalent. For $\mathbf{x} \in \mathbf{J}$,

$$\mathbf{x} \in \mathbf{A}^* \cap \mathbf{B}^* \Leftrightarrow (\mathbf{x} \in \mathbf{A}^* \text{ and } \mathbf{x} \in \mathbf{B}^*) \Leftrightarrow (\mathbf{x} \perp \mathbf{A} \text{ and } \mathbf{x} \perp \mathbf{B}) \Leftrightarrow \mathbf{x} = 0.$$

Thus from Eq. (1.11), $\mathbf{A}^* \wedge \mathbf{B}^* = \operatorname{span}(\mathbf{A}^*, \mathbf{B}^*)$. The right sides are equivalent.

Eq. (4.2) shows that the dual representation (Sec. 1.4) of the meet is $\mathbf{A}^* \wedge \mathbf{B}^*$.

4.1.4. Projective geometry. The homogeneous model enables us to study projective geometry algebraically in a coordinate-free manner. The blades of $\mathbb{G}^{n+1}$ represent the points, lines, ... of the projective space $\mathbf{P}^n$. Formulating projective geometry within GA should help to integrate it with the rest of mathematics.

In our visualization of the points of the homogeneous model of a plane, Fig. 8, a vector of $\mathbb{G}^3$ orthogonal to e does not intersect the embedded $\mathbb{R}^2$, and thus does not represent a point in the plane. For $\mathbf{P}^2$ it represents a point on the line at infinity.

GA provides simple algebraic tests for collinearity and concurrency in the projective plane $\mathbf{P}^2$:

- p, q, r collinear $\Leftrightarrow p \wedge q \wedge r = 0$. This is easy to see: $p \wedge q$ represents the line through p and q. From Eq. (1.11), r is on this line if and only if $p \wedge q \wedge r = 0$.
- P, Q, R concurrent $\Leftrightarrow \langle P\,Q\,R \rangle_0 = 0$. This is a bit harder. For the bivectors P, Q, and R, we have $(P^* \cdot Q) \cdot R^* = \langle P^* Q \rangle_{2-1} \cdot R^* = -\langle P\,Q\,R \rangle_0$. Thus from Eqs. (4.1) and (1.12),

  P, Q, R concurrent $\Leftrightarrow (P \vee Q) \wedge R = 0 \Leftrightarrow (P^* \cdot Q) \cdot R^* = 0 \Leftrightarrow \langle P\,Q\,R \rangle_0 = 0$.

Fig. 10 illustrates *Desargues' Theorem*, perhaps the most celebrated theorem of projective geometry: Given coplanar triangles abc and $a'b'c'$, construct P, Q, R and p, q, r. Then P, Q, R concurrent $\Leftrightarrow p$, q, r collinear.

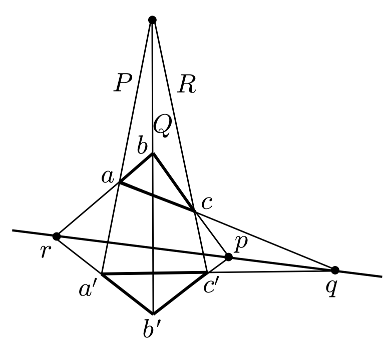


Fig. 10: Desargues theorem.

We indicate an algebraic proof of the theorem. In Fig. 10 we have $P = a \wedge a'$, with similar expressions for Q and R. Also, $p = (b \wedge c) \vee (b' \wedge c')$, with similar expressions for q and r.

Let $J = a \wedge b \wedge c$ and $J' = a' \wedge b' \wedge c'$. Then J and J' are nonzero multiples of the pseudoscalar. Thus JJ' is a nonzero scalar.

Desargues' theorem is, from the tests for collinearity and concurrency above, an immediate corollary of the identity $(p \wedge q \wedge r)^* = JJ' \langle P\,Q\,R \rangle_0$, which applies to *all* pairs of triangles. The identity is thus a generalization of the theorem. Unfortunately its proof is too involved to give here.

The geometric algebra $\mathbb{G}^n$ models two geometries: n-dimensional Euclidean space and $(n-1)$-dimensional projective space.

As we have seen, the homogeneous model is useful. But lengths and angles are not straightforward in the model. There are other problems. For example, the inverse of the representation of a vector is not the representation of the inverse of the vector.

4.2 The Conformal Model

The conformal model is the most powerful geometric algebra known for Euclidean geometry. It represents points, lines, planes, circles, and spheres as blades in the algebra. It represents the Euclidean group in a simple way as a group of automorphisms of the algebra. The model also unifies Euclidean, hyperbolic, and elliptic geometries.

4.2.1. The conformal model. Extend $\mathbb{R}^n$ with vectors e_+ and e_- orthogonal to $\mathbb{R}^n$, and satisfying $e_\pm^2 = \pm 1$. Then we have $\mathbb{R}^{n+1,1}$, and with it $\mathbb{G}^{n+1,1}$. (See Sec. 2.4.1 for indefinite metrics.) More useful than $e_\pm$ are $e_\mathrm{o} = \frac{1}{2}(e_- - e_+)$ and $e_\infty = e_- + e_+$. Then $e_\mathrm{o}^2 = e_\infty^2 = 0$ and $e_\mathrm{o} \cdot e_\infty = -1$.

The conformal model represents the point at the end of the vector $\mathbf{p} \in \mathbb{R}^n$ with

$$p = e_\mathrm{o} + \mathbf{p} + \frac{1}{2}\mathbf{p}^2 e_\infty \in \mathbb{R}^{n+1,1} \tag{4.3}$$

and its nonzero scalar multiples. The vector p is *normalized*: the coefficient of e_o is 1.

You can check that p is *null*, $p^2 = p \cdot p = 0$, and more generally that

$$p \cdot q = -\frac{1}{2}(\mathbf{p} - \mathbf{q})^2. \tag{4.4}$$

Append ∞, the *point at infinity*, to $\mathbb{R}^n$, and represent it with with e_∞. We now have an algebraic representation of $\bar{\mathbb{R}}^n \equiv \mathbb{R}^n \cup \infty$.

4.2.2. Representing geometric objects. For simplicity we work in 3D.

Spheres. Equation (4.4) immediately gives an equation of the sphere with center $\mathbf{c}$ and radius ρ: $x \cdot c = -\frac{1}{2}\rho^2$. This is equivalent to $x \cdot (\, c - \frac{1}{2}\rho^2 e_\infty) = 0$. Thus the *vector* $\sigma = c - \frac{1}{2}\rho^2 e_\infty$ is a dual representation of the *sphere*! The point c can be thought of as a sphere with radius $\rho = 0$.

We rewrite σ in terms of $\mathbf{c}$ and a point $\mathbf{p}$ on the sphere:

$$\sigma = c - \frac{1}{2}\rho^2 e_\infty = -(p \cdot e_\infty)\, c + (p \cdot c)\, e_\infty = p \cdot (c \wedge e_\infty), \tag{4.5}$$

where the last equality is an important general identity for vectors in GA.

The 4–blade $S = \sigma^*$ is a direct representation of the sphere. A direct representation can also be constructed from four points on the sphere: $S = p \wedge q \wedge r \wedge s$. Then the center and radius can be read off from $\alpha S^* = c - \frac{1}{2}\rho^2 e_\infty$, where α normalizes S^*.

A straightforward calculation shows that the inner product of the point $p = e_\mathrm{o} + \mathbf{p} + \frac{1}{2}\mathbf{p}^2 e_\infty$ and the sphere $\sigma = c - \frac{1}{2}\rho^2 e_\infty$ satisfies $-2p \cdot \sigma = (\mathbf{p} - \mathbf{c})^2 - \rho^2$. Thus p is inside, on, or outside σ according as $p \cdot \sigma$ is $> 0,\ = 0,$ or < 0, respectively.

Lines. A direct representation of the line through points $\mathbf{p}$ and $\mathbf{q}$ is $p \wedge q \wedge e_\infty$. To prove this, let x represent the point $\mathbf{x}$. Then:

$$\begin{aligned} x \wedge (p \wedge q \wedge e_\infty) = 0 \ &\Leftrightarrow\ x = ap + bq + ce_\infty \\ &\Leftrightarrow\ x = (a+b)e_\mathrm{o} + (a\mathbf{p} + b\mathbf{q}) + c' e_\infty \\ &\Leftrightarrow\ \mathbf{x} = \frac{a}{a+b}\,\mathbf{p} + \frac{b}{a+b}\,\mathbf{q}, \quad \frac{a}{a+b} + \frac{b}{a+b} = 1. \end{aligned}$$

The representation does not determine $\mathbf{p}$ or $\mathbf{q}$. But it does determine the distance between them: $(p \wedge q \wedge e_\infty)^2 = |\mathbf{p} - \mathbf{q}|^2$.

Planes. The representation $p \wedge q \wedge e_\infty$ of lines gives a test: $\mathbf{m}$, $\mathbf{p}$, $\mathbf{q}$ collinear $\Leftrightarrow$ $m \wedge (p \wedge q \wedge e_\infty) = 0$. If they are not collinear, then they determine a plane. The direct representation of the plane is $m \wedge p \wedge q \wedge e_\infty$, a "sphere through infinity". The area A of the triangle with vertices $\mathbf{m}$, $\mathbf{p}$, and $\mathbf{q}$ is given by $(m \wedge p \wedge q \wedge e_\infty)^2 = 4A^2$.

A dual representation of the plane through the point $\mathbf{p}$ and orthogonal to $\mathbf{n}$ is the vector $\pi = \mathbf{n} + (\mathbf{p} \cdot \mathbf{n})e_\infty$. This follows from $x \cdot \pi = (\mathbf{x} - \mathbf{p}) \cdot \mathbf{n}$ and the VA equation of the plane $(\mathbf{x} - \mathbf{p}) \cdot \mathbf{n} = 0$. A different form follows from the identity used in Eq. (4.5):

$$\pi = \mathbf{n} + (\mathbf{p} \cdot \mathbf{n})e_\infty = -(p \cdot e_\infty)\, \mathbf{n} + (p \cdot \mathbf{n})\, e_\infty = p \cdot (\mathbf{n} \wedge e_\infty). \tag{4.6}$$

If instead we are given $\mathbf{n}$ and the distance d of the plane to the origin, then the VA equation $\mathbf{x} \cdot \mathbf{n} = d$ gives the dual representation $\pi = \mathbf{n} + de_\infty$. From this, $d = -\pi \cdot e_0$.

Circles. A direct representation of the circle through $\mathbf{p}$, $\mathbf{q}$, and $\mathbf{r}$ is $p \wedge q \wedge r$. We can think of the lines $p \wedge q \wedge e_\infty$ above as circles through infinity.

We determine the dual representation of the circle with center $\mathbf{c}$, radius ρ, and normal $\mathbf{n}$. The circle is the intersection of the dual sphere $c - \frac{1}{2}\rho^2 e_\infty$ (Eq. (4.5)) and the dual plane $c \cdot (\mathbf{n} \wedge e_\infty)$ (Eq. (4.6)). From the dual representation of the meet, Eq. (4.2), the dual representation of the circle is the bivector $(c - \frac{1}{2}\rho^2 e_\infty) \wedge (\, c \cdot (\mathbf{n} \wedge e_\infty)\,)$.

4.2.3. Representing geometric operations. Now that we know how to represent Euclidean objects in the conformal model, we turn to representing geometric operations on the objects.

Let O be an orthogonal transformation of $\mathbb{R}^{n+1,1}$. Then $p \cdot p = 0 \Rightarrow \mathsf{O}(p) \cdot \mathsf{O}(p) = 0$. It is not difficult to show that every null vector of $\mathbb{G}^{n+1,1}$ is of the form λe_∞ or λp for some $\mathbf{p} \in \mathbb{R}^n$. Thus O induces a map of $\bar{\mathbb{R}}^n$ to itself. Such maps form the conformal group of $\bar{\mathbb{R}}^n$. (They preserve angles, but not necessarily orientations.)

The conformal group is generated (redundantly) by the following transformations:

- **Rotations.** A rotation around the origin by angle $\mathbf{i}\theta$ is represented in the conformal model just as in the vector space model: $p \to \mathrm{e}^{-\mathbf{i}\theta/2} p\, \mathrm{e}^{\mathbf{i}\theta/2}$. (Eq. (2.4).)

- **Translations.** The translation $\mathbf{p} \to \mathbf{p} + \mathbf{a}$ is represented by $p \to \mathrm{e}^{-\mathbf{a}e_\infty/2} p \mathrm{e}^{\mathbf{a}e_\infty/2}$. Note that since $e_\infty\, \mathbf{a} = -\mathbf{a}\, e_\infty$ and $e_\infty^2 = 0$, $\mathrm{e}^{\pm\mathbf{a}e_\infty/2} = 1 \pm \mathbf{a}\, e_\infty/2$ (exactly).

- **Reflections.** A reflection of the point p in a hyperplane with normal vector $\mathbf{n}$ is represented in the conformal model just as in the vector space model: $p \to -\mathbf{n}\, p\, \mathbf{n}^{-1}$. (Theorem 1.5.11b.)

- **Inversions.** The inversion $\mathbf{p} \to \mathbf{p}^{-1}$ is represented in the conformal model by a reflection in the hyperplane normal to e_+: $p \to -e_+\, p\, e_+$. To normalize the result, divide by the coefficient of e_0 (which is $\mathbf{p}^2$).

- **Dilations.** The dilation $\mathbf{p} \to \alpha\mathbf{p}$ is represented in the conformal model by $p \to \mathrm{e}^{-\ln(\alpha)E/2} p\, \mathrm{e}^{\ln(\alpha)E/2}$, where $E = e_\infty \wedge e_\mathrm{o}$. Note that $\mathrm{e}^{\beta E} = \cosh\beta + E \sinh\beta$. To normalize the result, divide by the coefficient of e_0 (which is $1/\alpha$).

The subgroup of the conformal group fixing e_∞ is the Euclidean group of $\mathbb{R}^n$. For rotations, translations, and reflections fix e_∞, while inversions and dilations do not.

4.2.4. Conformal transformations as automorphisms. As we have seen, the representation of a conformal transformation of $\mathbb{R}^n$ is constructed simply from its geometric description. The representations are of the form $p \to \pm V p V^{-1}$. They extend to automorphisms of the entire geometric algebra. The automorphisms are of the form $M \to \pm V M V^{-1}$. The automorphisms preserve the geometric, inner, and outer products and grades. They compose simply, as do rotations in the vector space model. (Section 2.2.2). The following examples show how all this can be used to advantage.

The rotations in Sec. 4.2.3 are around the origin. To rotate an object around $\mathbf{a}$, translate it by $-\mathbf{a}$, rotate it around the origin, and translate it back by $\mathbf{a}$. Thus a rotation around $\mathbf{a}$ is represented by $T_\mathbf{a} \mathrm{e}^{-\mathbf{i}\theta/2} T_\mathbf{a}^{-1}$, where $T_\mathbf{a} = \mathrm{e}^{-\mathbf{a}e_\infty/2}$. This says that rotations translate in the same way as geometric objects. There is more. You can quickly check that $T_\mathbf{a} \mathrm{e}^{-\mathbf{i}\theta/2} T_\mathbf{a}^{-1} = \mathrm{e}^{-T_\mathbf{a}(\mathbf{i}\theta)T_\mathbf{a}^{-1}/2}$ using the power series expansion of the exponential. This says that the angle $T_\mathbf{a}(\mathbf{i}\theta)T_\mathbf{a}^{-1}$ specifies the rotation around $\mathbf{a}$. It is the translation by $\mathbf{a}$ of the angle $\mathbf{i}\theta$ specifying the rotation around 0.

We compute the angle θ between intersecting lines ℓ_1 and ℓ_2. Translate the lines so that their intersection is at the origin. They now have representations $\ell_1' = e_\mathrm{o} \wedge p_1 \wedge e_\infty$ and $\ell_2' = e_\mathrm{o} \wedge p_2 \wedge e_\infty$. Because of the simple form of ℓ_1' and ℓ_2', it is easy to show that

$$(\ell_1' \cdot \ell_2')/|\ell_1'|\,|\ell_2'| = (\mathbf{p}_1 \cdot \mathbf{p}_2)/|\mathbf{p}_1|\,|\mathbf{p}_2| = \cos\theta.$$

Since the translation preserves inner products and norms, we also have $(\ell_1 \cdot \ell_2)/|\ell_1|\,|\ell_2| = \cos\theta$.

Now let σ_1 and σ_2 be coplanar circles intersecting at two points. (Circles intersecting tangentially require a separate analysis.) Translate the circles so that an intersection point is at the origin. Now perform an inversion. This conformal transformation maps the circles to straight lines: $p \wedge q \wedge e_\mathrm{o} \to p' \wedge q' \wedge e_\infty$. Thus the angle between the circles is given by the same formula as for the lines: $\cos\theta = (\sigma_1 \cdot \sigma_2)/|\sigma_1|\,|\sigma_2|$.

4.2.5. Noneuclidean geometry. The conformal model can be used for hyperbolic and elliptic geometry [1]. Let $\mathcal{B}$ be the unit ball centered at the origin of $\mathbb{R}^n$.

$\mathcal{B}$ provides a model for hyperbolic geometry:

- The hyperbolic line through $\mathbf{p}, \mathbf{q} \in \mathcal{B}$ is $p \wedge q \wedge e_+$.
  (This is the Euclidean circle through $\mathbf{p}$ and $\mathbf{q}$ intersecting $\partial\mathcal{B}$ orthogonally.)
- The hyperbolic distance d between $\mathbf{p}$ and $\mathbf{q}$ satisfies $p \cdot q = -2\sinh^2(d/2)$.
  (The coefficient of e_+ in p and q must be normalized to 1.)
- The hyperbolic group is the subgroup of $O(n+1,1)$ fixing e_+ .

$\mathcal{B}$ provides a model for elliptic geometry:

- The elliptic line through $\mathbf{p}, \mathbf{q} \in \mathcal{B}$ is $p \wedge q \wedge e_-$.
  (This is the Euclidean circle through $\mathbf{p}$ and $\mathbf{q}$ intersecting $\partial\mathcal{B}$ at antipodes.)
- The elliptic distance d between $\mathbf{p}$ and $\mathbf{q}$ satisfies $p \cdot q = -2\sin^2(d/2)$.
  (The coefficient of e_- in p and q must be normalized to -1.)
- The elliptic group is the subgroup of $O(n+1,1)$ fixing e_- .

5 Useful Identities

$A \cdot (B \cdot C) = (A \wedge B) \cdot C\,.$

$(\mathbf{A} \cdot B) \cdot \mathbf{C} = \mathbf{A} \wedge (B \cdot \mathbf{C})$ if $\mathbf{A} \subseteq \mathbf{C}$

$B \wedge A = (-1)^{jk}(A \wedge B)\,.$ (A is a j-vector and B is a k-vector.)
$\quad \mathbf{a} \wedge A \wedge \mathbf{b} = -\mathbf{b} \wedge A \wedge \mathbf{a}\,.$
$\quad\quad \mathbf{a} \wedge A \wedge \mathbf{a} = 0\,.$

$(\mathbf{a} \wedge \mathbf{b}) \cdot (\mathbf{c} \wedge \mathbf{d}) = (\mathbf{a} \cdot \mathbf{d})(\mathbf{b} \cdot \mathbf{c}) - (\mathbf{a} \cdot \mathbf{c})(\mathbf{b} \cdot \mathbf{d})\,.$

$\mathbf{a} \cdot (BC) = (\mathbf{a} \cdot B)\, C + (-1)^k B(\mathbf{a} \cdot C)\,.$ (B is a k-vector.)
$\quad \mathbf{a} \cdot (\mathbf{b}\mathbf{c}) = (\mathbf{a} \cdot \mathbf{b})\, \mathbf{c} - (\mathbf{a} \cdot \mathbf{c})\, \mathbf{b}\,.$
$\quad \mathbf{a} \cdot (\mathbf{b}_1 \mathbf{b}_2 \cdots \mathbf{b}_k) = \sum_r (-1)^{r+1} (\mathbf{a} \cdot \mathbf{b}_r)\, \mathbf{b}_1 \cdots \check{\mathbf{b}}_r \cdots \mathbf{b}_k\,.$ ($\check{\mathbf{b}}_r$ means omit $\mathbf{b}_r$)

$\mathbf{a} \cdot (B \wedge C) = (\mathbf{a} \cdot B) \wedge C + (-1)^k B \wedge (\mathbf{a} \cdot C)\,.$ (B is a k-vector.)
$\quad \mathbf{a} \cdot (\mathbf{b} \wedge \mathbf{c}) = (\mathbf{a} \cdot \mathbf{b})\, \mathbf{c} - (\mathbf{a} \cdot \mathbf{c})\, \mathbf{b}\,.$
$\quad \mathbf{a} \cdot (\mathbf{b}_1 \wedge \mathbf{b}_2 \wedge \cdots \wedge \mathbf{b}_k) = \sum_r (-1)^{r+1} (\mathbf{a} \cdot \mathbf{b}_r)\, \mathbf{b}_1 \wedge \cdots \wedge \check{\mathbf{b}}_r \wedge \cdots \wedge \mathbf{b}_k\,.$
Thus for a blade $\mathbf{B} = \mathbf{b}_1 \wedge \mathbf{b}_2 \wedge \cdots \wedge \mathbf{b}_k,\ \mathbf{a} \cdot \mathbf{B} = 0 \;\Leftrightarrow\; \mathbf{a} \cdot \mathbf{b}_r = 0$ for all r.

$\mathbf{a} \wedge (B \cdot C) = (\mathbf{a} \cdot B) \cdot C + (-1)^k B \cdot (\mathbf{a} \wedge C)\,.$ (B is a k-vector.)

$(\mathbf{A} \cdot \mathbf{B})^* = \mathbf{A} \wedge \mathbf{B}^*.$ (Duality)
$(\mathbf{A} \wedge \mathbf{B})^* = \mathbf{A} \cdot \mathbf{B}^*.$ (Duality)

Take * with respect to the join of $\mathbf{A}$ and $\mathbf{B}$. Then
$\mathbf{A} \vee \mathbf{B} = \mathbf{A}^* \cdot \mathbf{B}.$
$(\mathbf{A} \vee \mathbf{B})^* = \mathbf{A}^* \wedge \mathbf{B}^*.$

6 Further Study

Web Sites

David Hestenes
http://geocalc.clas.asu.edu/

Cambridge University Geometric Algebra Research Group
http://www.mrao.cam.ac.uk/~clifford/

University of Amsterdam Geometric Algebra Website
http://www.science.uva.nl/ga/

Books

A. Macdonald, *Linear and Geometric Algebra*
Web page: http://faculty.luther.edu/~macdonal/laga/
Textbook for the sophomore year linear algebra course includes geometric algebra.

A. Macdonald, *Vector and Geometric Calculus*
Web page: http://faculty.luther.edu/~macdonal/vagc/
Textbook for the sophomore year vector calculus course includes geometric calculus.

C. Doran and A. Lasenby, *Geometric Algebra for Physicists*
(Cambridge University Press, 2003).
Web page: http://www.mrao.cam.ac.uk/~cjld1/pages/book.htm
Development of geometric algebra and calculus for applications to physics.
Noneuclidean geometry.

D. Hestenes, *New Foundations for Classical Mechanics*
(Kluwer Academic Publishers, 1999).
Web page: http://geocalc.clas.asu.edu/html/NFCM.html
Textbook suitable for courses at an intermediate level.

L. Dorst, D. Fontijne, S. Mann, *Geometric Algebra for Computer Science*:
An Object-Oriented Approach to Geometry (Morgan Kaufmann, 2$^{\text{nd}}$ printing, 2009)
Web page: http://www.geometricalgebra.net/
Development of geometric algebra for applications to computer science.

Christian Perwass, *Geometric Algebra with Applications in Engineering*
(Springer, 2008)

D. Hestenes and G. Sobczyk, *Clifford Algebra to Geometric Calculus*
(Kluwer Academic Publishers, 1984).
Web page: http://geocalc.clas.asu.edu/html/CA_to_GC.html
The bible. Not for the faint of heart.

Papers on the Web

Elementary.

A unified mathematical language for physics and engineering in the 21st century
http://www.mrao.cam.ac.uk/~clifford/publications/abstracts/dll_millen.html
Synopsis of Geometric Algebra
http://geocalc.clas.asu.edu/html/NFMP.html
Geometric Algebra: a computational framework for geometrical applications, I & II
Performance and elegance of five models of 3D Euclidean geometry
http://www.science.uva.nl/ga/publications/CGnA.html
Physical Applications of Geometric Algebra
This is a preliminary version of Doran and Lasenby's book.
http://www.mrao.cam.ac.uk/~clifford/ptIIIcourse/
An Elementary Construction of the Geometric Algebra
http://faculty.luther.edu/~macdonal

Advanced.

Clifford algebra, geometric algebra, and applications
http://arxiv.org/abs/0907.5356
Applications of Clifford's Geometric Algebra
http://arxiv.org/abs/1305.5663

History.

On the Evolution of Geometric Algebra and Geometric Calculus
http://geocalc.clas.asu.edu/html/Evolution.html
A unified mathematical language for physics and engineering in the 21st century
http://www.mrao.cam.ac.uk/~clifford/publications/abstracts/dll_millen.html
Grassmann's Vision
http://geocalc.clas.asu.edu/html/GeoAlg.html#Grassmann

Polemics.

Oersted Medal Lecture 2002: Reforming the mathematical language of physics
http://geocalc.clas.asu.edu/html/Oersted-ReformingTheLanguage.html
Mathematical Viruses
http://geocalc.clas.asu.edu/html/GeoAlg.html
Unified Language for Mathematics and Physics
http://geocalc.clas.asu.edu/html/GeoCalc.html
The inner products of geometric algebra
http://www.science.uva.nl/ga/publications/index.html
Differential Forms in Geometric Calculus
http://geocalc.clas.asu.edu/pdf-preAdobe8/DIF_FORM.pdf

Calculus.

Multivector Calculus
Multivector Functions
http://geocalc.clas.asu.edu/html/GeoCalc.html
Geometric Calculus
http://geocalc.clas.asu.edu/html/GeoCalc.html

Spacetime and Electromagnetism.

Spacetime Physics with Geometric Algebra
http://geocalc.clas.asu.edu/html/Oersted-ReformingTheLanguage.html
SpaceTime Calculus
http://geocalc.clas.asu.edu/html/STC.html

Quantum Mechanics.
Various papers
http://geocalc.clas.asu.edu/html/GAinQM.html
Linear Algebra and Geometry.
The Design of Linear Algebra and Geometry
Universal Geometric Algebra
Projective Geometry with Clifford Algebra
http://geocalc.clas.asu.edu/html/GeoAlg.html
(Interactively) Exploring the conformal model of 3D Euclidean geometry
http://www.science.uva.nl/ga/tutorials/CGA/index.html
Conformal Geometry, Euclidean Space and Geometric Algebra
Paper dll_sheffield.ps.gz at
http://www.mrao.cam.ac.uk/~clifford/publications/ps/
Old Wine in New Bottles: A new algebraic framework for computational geometry
http://geocalc.clas.asu.edu/html/ComputationalGeometry.html
A Unified Algebraic Framework for Classical Geometry
http://geocalc.clas.asu.edu/html/UAFCG.html
Lie Groups as Spin Groups
http://geocalc.clas.asu.edu/html/GeoAlg.html
A Covariant Approach to Geometry using Geometric Algebra
Includes noneuclidean geometry.
http://www-sigproc.eng.cam.ac.uk/ga/index.php?title=A_Covariant_Approach_to_Geometry_using_Geometric_Algebra
Geometric Computing in Computer Graphics using Conformal Geometric Algebra
Two interesting applications to computer graphics.
http://www.gris.informatik.tu-darmstadt.de/~dhilden/CLUScripts/CandG_PrePrint.pdf

Software

GAViewer, software and tutorials.
http://www.science.uva.nl/ga/viewer/gaviewer_download.html
http://www.science.uva.nl/ga/tutorials/index.html

Maple software.
http://www.mrao.cam.ac.uk/~clifford/pages/software.htm
I have fixed a few bugs and added several procedures, including one for the inner product used here. It works on Maple V and perhaps higher versions.
http://faculty.luther.edu/~macdonal

CluCalc.
http://www.clucalc.info/

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http://www.science.uva.nl/ga/publications/content_publications.html#leo_inner.
Another version:
http://www.science.uva.nl/ga/publications/content_talks.html#inner_talk

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http://geocalc.clas.asu.edu/html/ComputationalGeometry.html.

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http://planetmath.org/encyclopedia/RodriguesRotationFormula.html.

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