Strömgren Spheres
and
Recombination Lines

Now that we have looked at the equations that govern photoionization and recombination, we are ready to consider the relations between hot stars and the nebulae that surround them. The energy radiated from a unit area of a star's surface is written as $\pi F(\nu)$, where the flux $F(\nu)$ may be obtained from a model atmosphere calculated for a specific temperature, surface gravity and chemical composition. To get the basic ideas, we will replace F with the blackbody Planck function $B(\nu,T)$. We can then divide by the photon energy $h\nu$ and integrate over all frequencies from the ionization threshold to infinity and multiply by the surface area of the star to obtain $Q(H^0)$, the number of ionizing photons the star emits.

$$Q(H^0) = 4\pi R^2 \int_{\nu_0}^{\infty} \frac{\pi B_\nu(T)}{h\nu}\, d\nu \quad \text{,where} \quad B_\nu(T) = \frac{2h\nu^3}{c^2} \frac{1}{e^{h\nu/kT} - 1}$$

Here ν_0 is the threshold frequency ($h\nu_0$ is the ionization potential) and R the radius of the star. To do the integral, we introduce the variable $x=h\nu/kT$ and obtain

$$x_0 = \frac{h\nu_0}{kT}, \quad \nu = \frac{kT}{h}x, \quad d\nu = \frac{kT}{h}dx \quad \Longrightarrow \quad Q(H^0) = 4\pi R^2 \, \frac{2\pi}{c^2} \left(\frac{kT}{h}\right)^3 \int_{x_0}^{\infty} \frac{x^2 \, dx}{e^x - 1}$$

The integral can be evaluated by expanding in a binomial series as follows

$$\mathcal{I}(x_0) = \int_{x_0}^{\infty} \frac{x^2 \, dx}{e^x - 1} = \int_{x_0}^{\infty} \frac{x^2 e^{-x} dx}{1 - e^{-x}} = \int_{x_0}^{\infty} x^2 e^{-x} \left[1 + e^{-x} + e^{-2x} + e^{-3x} + \cdots\right] dx$$

The last integrals are elementary so we have that

$$\mathcal{I}(x_0) \;=\; \sum_{n=1}^{\infty} \frac{(nx_0)^2 + 2nx_0 + 2}{n^3} \; e^{-nx_0}$$

Putting in all the constants, we finally obtain

$$Q(H^0) = 3.844 \times 10^{45}\ t^3\ \mathcal{I}(x_0)\ (R/R_\odot)^2$$

$$\text{where } t = 10^{-4}T \quad \text{and} \quad x_0 = 15.78/t$$

Here are some results compared to the values from model atmospheres. Not too bad.

Spectral Type	Effective Temperature	Radius (solar units)	$\mathcal{I}(x_0)$	Blackbody $Q(H^0)$	Model Atmosphere $Q(H^0)$
B5 V	15,200 K	3.9	0.00404	8.30×10^{44}	–
B0 V	30,000 K	7.4	0.209	1.19×10^{48}	1.45×10^{48}
O5 V	42,000 K	12	0.557	2.28×10^{49}	3.39×10^{49}

In the same way we can estimate the number of photons able to doubly ionize helium, Q(He+), by simply increasing the threshold energy by a factor of 4. We would find that we would need T in excess of ~80,000 K to get substantial values of Q(He+).

Now suppose our star is embedded in an H cloud of uniform density. The ionizing radiation will form an H+ region around the star. Strömgren (1939) showed that there would be a sharp transition from fully ionized to neutral gas -- the H+ region would be a sphere -- the "Strömgren sphere" -- wherein the rate of constant recombinations will balance the input of photons given by $Q(H^0)$. This allows us to write down an equation for the radius of the H+ region:

$$Q(H^0) = \frac{4\pi}{3}\, r_s^3\, \alpha_B\, n_e n(H^+) \quad \Longrightarrow \quad r_s = \left\{ \frac{3\, Q(H^0)}{4\pi\, n_H^2\, \alpha_B} \right\}^{1/3}$$

As an example, consider a Ströngren sphere around the B0V star in the table. We will use the model atmosphere value of $Q(H^0)$, 1.45e48 photon/s. Let us assume a hydrogen density of 500 per cm³. Further, we evaluate the recombination coefficient at 9000 K, a typical value for H+ nebulae. Plugging these numbers into our equation gives a radius of about half a parsec:

$$r_s = \left\{ \frac{3 \cdot 1.45 \times 10^{48}}{4\pi \cdot 500^2 \cdot 2.77 \times 10^{-13}} \right\}^{1/3} = 1.71 \times 10^{18} \; cm = 0.554 \; pc$$

Real nebulae are of course not uniform in density, but the concept of the Strömgren sphere does capture the idea of sharply bounded H+ regions and shows how the size is related to the density of the gas, as well as to the spectral type of the exciting star(s).

The planetary nebula IC 418. The ionized region seen here is surrounded by neutral gas, so this is essentially a Strömgren sphere. The temperature of the central star is about 35,000 K.

Recombination Lines

The prominent emission lines of H and He seen in the spectra of gaseous nebulae are the result of recombinations. For H the rate of recombinations to any energy level is a weak function of temperature and these rates have been calculated and tabulated. Nevertheless, evaluating the emission in a particular spectral line is complex. If the emission line is produced by a jump from level n' to n (e.g. , 4→2 for Hβ) we need not only the rate of recombinations to n', but also the rate to all levels n''>n', since those levels may feed n' by n''→n' jumps. In fact, it is necessary to solve the cascade problem, where we follow all the possible downward jumps, feeding n' by any path. In addition, we need to know the branching probabilities, i.e., the fraction of electrons arriving at n' that then jump to n, rather than some other lower level (i.e., 4→2 rather than 4→3 or 4→1). Fortunately, this has been worked out long ago and tabulated. Even better, it turns out that the results do not depend greatly on the gas temperature.

One result of this is that the relative intensities of the H lines are known. The H lines in the optical part of the spectrum are known as the Balmer series, and the way in which the intensities of these lines decrease as we go up the series (i.e., Hα, Hβ, Hγ, ...) is called the Balmer decrement. If we measure line ratios that deviate from these theoretical ratios, say if Hα were 4 times stronger than Hβ, we could usually conclude that this was due to interstellar reddening by dust, and correct all the other line intensities accordingly. Another possibility would be that there is a contribution to the Hα line by collisional excitation -- a rare occurrence.

Just as the recombination coefficient gives the rate of all recombinations, we can define an effective recombination coefficient for Hβ (or any other line) which gives the rate of recombinations that result in the emission of an Hβ photon:

$$\alpha_{eff}(H\beta) = 3.03 \times 10^{-14} \; t^{-0.874} \quad cm^3 s^{-1}$$

Energy Levels of the H^0 atom

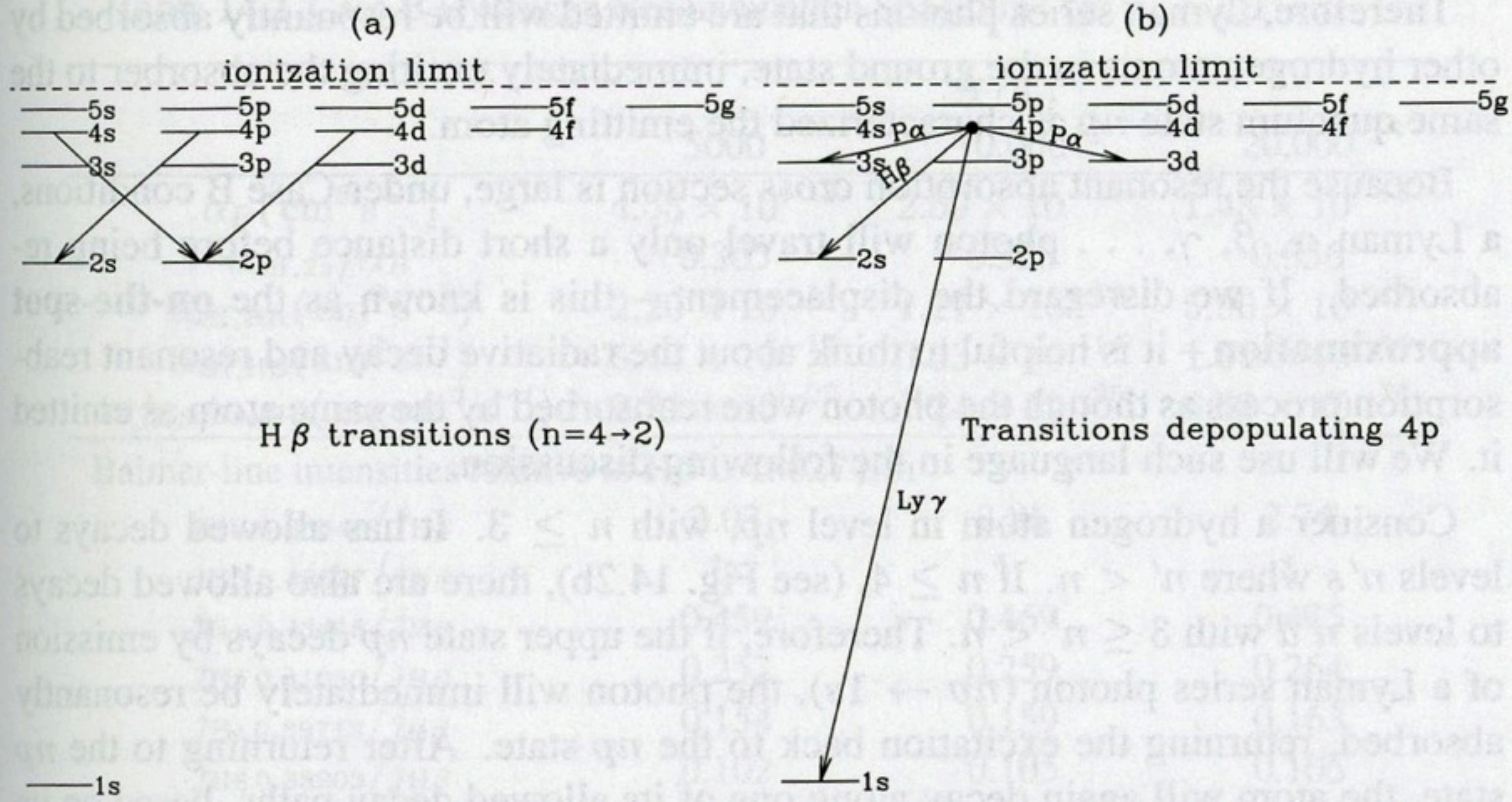

Figure 14.2 (a) The three transitions contributing to Hβ; (b) Allowed radiative transitions depopulating the 4p level.

Table 14.2 Case B Hydrogen Recombination Spectrum[a] for $n_e = 10^3$ cm^{-3}

	T(K)		
	5000	10,000	20,000
α_B(cm^3 s^{-1})	4.53×10^{-13}	2.59×10^{-13}	1.43×10^{-13}
$\alpha_{\rm eff,2s}/\alpha_B$	0.305	0.325	0.356
$\alpha_{\rm eff,H\alpha}$(cm^3 s^{-1})	2.20×10^{-13}	1.17×10^{-13}	5.96×10^{-14}
$\alpha_{\rm eff,H\beta}$(cm^3 s^{-1})	5.40×10^{-14}	3.03×10^{-14}	1.61×10^{-14}
$4\pi j_{\rm H\beta}/n_e n_p$(erg cm^3 s^{-1})	2.21×10^{-25}	1.24×10^{-25}	6.58×10^{-26}
Balmer-line intensities relative to Hβ 0.48627 μm			
$j_{\rm H\alpha\,0.65646}/j_{\rm H\beta}$	3.03	2.86	2.74
$j_{\rm H\beta\,0.48627}/j_{\rm H\beta}$	1.	1.	1.
$j_{\rm H\gamma\,0.43418}/j_{\rm H\beta}$	0.459	0.469	0.475
$j_{\rm H\delta\,0.41030}/j_{\rm H\beta}$	0.252	0.259	0.264
$j_{\rm H\epsilon\,0.39713}/j_{\rm H\beta}$	0.154	0.159	0.163
$j_{\rm H8\,0.38902}/j_{\rm H\beta}$	0.102	0.105	0.106
$j_{\rm H9\,0.38365}/j_{\rm H\beta}$	0.0711	0.0732	0.0746
$j_{\rm H10\,0.37990}/j_{\rm H\beta}$	0.0517	0.0531	0.0540

Since we measure the energy radiated in a spectral line, not the number of photons, we must multiply this rate by the photon energy to obtain the energy radiated in Hβ per unit volume:

$$4\pi j(H\beta) \;=\; n_e n(H^+)\; h\nu_\beta\; \alpha_{eff}(H\beta) \;=\; 1.24 \times 10^{-25}\; n_e n(H^+)\; t^{-0.874}$$

Here j(Hβ) is the energy emitted per steradian, so 4πj is the total emission per unit volume.

We can use this expression for j(Hβ) to do neat things. Consider the H+ region around the B0 star that we looked earlier. The total luminosity of this nebula in the Hβ line is given by 4π j(H) times the volume of the nebula:

$$L(H\beta) = \frac{4\pi}{3}\; r_s^3\; 4\pi j(H\beta)$$

$$L(H\beta) = \frac{4\pi}{3}(1.71 \times 10^{18})^3 \cdot 500^2 \cdot 1.24 \times 10^{-25} \cdot 0.9^{-0.874} = 7.12 \times 10^{35} \;\; erg\ s^{-1}$$

This seems like a lot of energy in one spectral line (the sun's luminosity is only 3.8e33), but consider that the luminosity of a B0 V star is >1e38 ergs/s, and that the nebula is capturing all the star's energy beyond 13.6 eV and converting it into line emission. Next, let's translate this into the flux we measure at the earth, F(Hβ). If the distance to the nebula is D, the flux we measure is the luminosity spread out over the surface area of a sphere of radius D centered on the nebula. Let's assume the nebula is at a distance of D = 1 kpc = 3.086e21 cm. Then we find

$$F(H\beta) \;=\; \frac{L(H\beta)}{4\pi D^2} \;=\; \frac{7.12 \times 10^{35}}{4\pi(3.086 \times 10^{21})^2} = 5.95 \times 10^{-9} \;\; erg\ cm^{-2}\ s^{-1}$$

In spite of what I said above, when dealing with modern instruments, it may be more convenient to have the flux in photon numbers. Since hν = 4.085e-12 at Hβ, we have

$$F(H\beta) \;=\; 1457 \;\; \text{photons cm}^{-2}\ \text{s}^{-1}$$

Such a flux is readily detected.

We can turn this problem around. From the observed flux, if we know the distance to the nebula, we have the luminosity L(Hβ). Then, noticing that both L(Hβ) and $Q(H^0)$ depend upon the integral over the volume of the nebula of n^2 times recombination coefficients that have a very similar temperature dependance, we take the ratio of coefficients out of the integral and thus find that

$$Q(H^0) = \frac{\alpha_B(T)}{\alpha_{eff}(H\beta)} \int_{vol} n_e n(H^+)\ \alpha_{eff}(H\beta)\ dV \ = \ 105.3\ D^2\ F(H\beta)$$

where both F and Q are in photons/sec. Thus the nebula is a photon counter, counting the far UV photons from the star and transmitting an optical signal to us.

Helium recombination lines

The other strong recombination lines is nebular spectra are due to recombinations and cascades through the energy levels of He^0 and, for highly excited gases, He+. Some of the transitions of He^0 are shown on the next page (the He+ levels look just like H^0). We see that He^0 is much more complicated, with the levels divided into singlet and triplet states that hardly communicate. About 3/4 of the recombinations go to the triplets. However, the ground state is a singlet, so the triplet recombinations and cascades end up stuck in an excited level with a lifetime of 2 hours (this is what is called a metastable state). The strongest line is the 10833 Å line (actually a triplet: 10832.1, 10833.2 and 10833.3 Å) in the near IR. Strong lines in the optical are 3890 Å, 4473 Å, 5877 Å and 6680 Å with relative intensities of 2.2, 1.0, 2.7 and 0.8, though these vary with T and density.

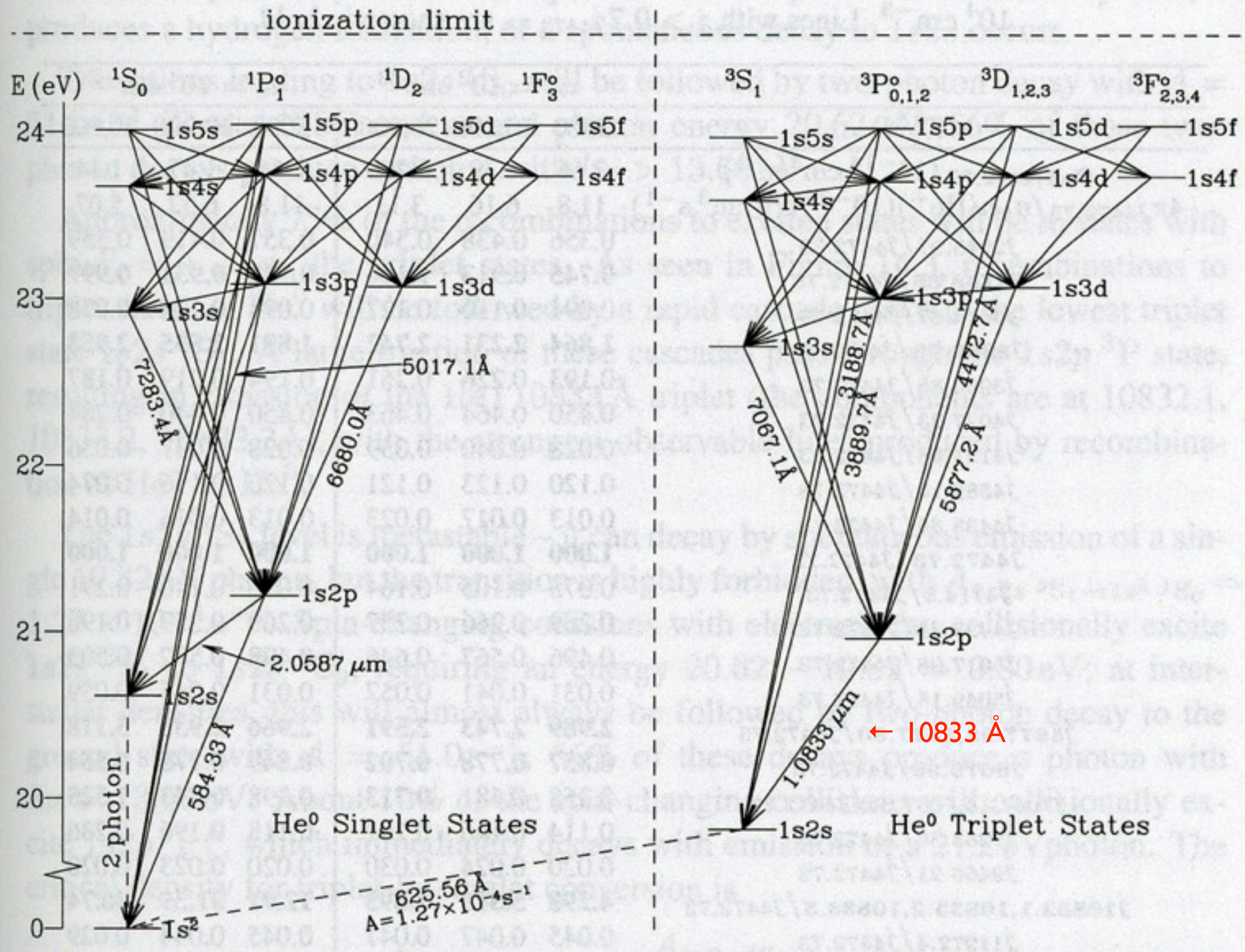

Figure 14.3 Radiative decay pathways for He^0 (see text). Selected lines are labeled by vacuum wavelength.