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ON SCHUSTER'S EMISSION-LINE MECHANISM

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ABSTRACT

Schuster's mechanism is reexamined for the case of a gray atmosphere, and the conditions under which emission can occur are given. No strong lines are expected to appear in emission, although this mechanism could contribute to emission in line wings under extreme conditions

I. INTRODUCTION

Coherent scattering in stellar atmospheres gives to the source function a non-LTE character which makes it possible for spectral lines to appear in emission, even though the *line* source function does not deviate from LTE. This possibility was first considered by Schuster (1905). The mechanism was studied in detail by Gebbie and Thomas (1968), who found that a necessary but not sufficient condition for emission was that the line occur at a wavelength greater than $\lambda = 9010$ Å $(10^{-4} T_0)^{-1}$, where T_0 is the surface temperature of the atmosphere. More restrictive conditions were obtained which depend on the atmospheric and line parameters. Their analysis, however, assumed that the Planck function $B_\nu(\tau)$ varied linearly with optical depth, and the slope and intercept of this linear approximation was obtained by evaluating $dB_\nu/d\tau$ at the surface. The Planck function in a gray atmosphere has, in fact, a monotonically decreasing slope as we go deeper into the atmosphere. If a linear approximation is used, it should represent the gross behavior of the Planck function over the region of the atmosphere where the emergent photons originate. Even in a discussion of strong lines, which do originate near the surface, it is necessary to consider the continuum, for it provides the reference level in the definition of an emission feature. Now the spectral distributions of the continuous radiation from two atmospheres having the same effective temperature but different ratios of scattering to absorption are not the same, even in the gray case where there is a unique $T(\tau)$ relation. The addition of scattering to the atmosphere can raise or lower the continuum. The continuum, however, originates at $\tau \simeq \frac{2}{3}$—or $\tau > \frac{2}{3}$ if the continuous absorption is small compared with the scattering, since then the thermalization length in the continuum increases, and photons can diffuse from deeper layers. Because the linear approximation which represents $B_\nu(\tau)$ near the surface is not adequate for the continuum, the criteria derived by Gebbie and Thomas are too stringent. We will first obtain improved criteria, and then consider whether conditions for emission have been relaxed enough to make the Schuster mechanism a practical possibility.

II. THE COHERENT-SCATTERING LINE SOURCE FUNCTION

We consider a coherent-scattering line source function of the form

$$S_l = \frac{J_\nu + \epsilon B_\nu}{1 + \epsilon} = (1 - L_l)J_\nu + L_l B_\nu . \tag{1}$$

This contains the LTE line source function as the special case $L_l = 1$. The total source function can be written as

$$S_\nu = (1 - L_\nu)J_\nu + L_\nu B_\nu , \tag{2}$$

where

$$L_\nu = \frac{r_\nu L_l + L_c}{1 + r_\nu}. \tag{3}$$

Here, $r_\nu = l_\nu/k$, $L_c = \kappa/k$, $k = \kappa + \sigma$, and l_ν, κ, and σ are the line absorption, the continuous pure absorption, and the continuous scattering, respectively. The optical depth in the line, τ_ν, is related to the continuum optical depth, τ_c, by $d\tau_\nu = (1 + r_\nu)d\tau_c$.

We assume that L_ν is not a function of optical depth. In the Eddington approximation the transfer equation is

$$\frac{1}{3}\frac{d^2 J_\nu}{d\tau_\nu^2} = L_\nu(J_\nu - B_\nu), \tag{4}$$

which can be solved by standard methods. With the surface boundary condition

$$\left[\frac{dJ_\nu}{d\tau_\nu}\right]_{\tau_\nu=0} = \tfrac{3}{4}F_\nu(0) = 3^{1/2}J_\nu(0), \tag{5}$$

we may write the solution for the emergent flux in the form

$$F_\nu(0) = \frac{4}{\sqrt{3}}\frac{L_\nu^{1/2}}{1 + L_\nu^{1/2}}\int_0^\infty B_\nu(\zeta)e^{-\zeta}d\zeta, \tag{6}$$

where

$$\zeta = (3L_\nu)^{1/2}\tau_\nu = [3(1 + r_\nu)(r_\nu L_l + L_c)]^{1/2}\tau_c. \tag{7}$$

Note that the character of the emergent radiation is affected by values of B_ν down to $\zeta \approx 1$, i.e., $\tau_\nu \sim (3L_\nu)^{-1/2}$.

We will consider a gray atmosphere with the temperature gradient

$$T(\tau_c) = 3^{1/8}T_0[\tau_c + q(\tau_c)]^{1/4}, \tag{8}$$

so that the exponent in the Planck function, $y = hc/\lambda kT$, is

$$y = y_0\{3^{1/2}[\tau_c + q(\tau_c)]\}^{-1/4}, \tag{9}$$

where $y_0 = hc/\lambda kT_0$.

For a line to appear in emission, it is necessary that $F_\nu(0)$ increase with an increase in line opacity:

$$dF_\nu(0)/dr_\nu > 0. \tag{10}$$

Taking the derivative of equation (6), we obtain

$$\frac{dF_\nu(0)}{dr_\nu} = \frac{4}{\sqrt{3}}\frac{L_\nu^{1/2}}{1 + L_\nu^{1/2}}\int_0^\infty B_\nu(\zeta)\left\{\frac{L_l - L_c}{2L_\nu(1 + L_\nu^{1/2})(1 + r_\nu)^2} - \frac{y}{4(1 - e^{-y})}\left[\frac{\tau_c(1 + dq/d\tau_c)}{\tau_c + q}\right]\frac{(1 + 2r_\nu)L_l + L_c}{2(1 + r_\nu)(r_\nu L_l + L_c)}\right\}e^{-\zeta}d\zeta. \tag{11}$$

It is clear that this expression cannot be positive unless $L_l > L_c$, so this is a necessary condition for emission. This condition is equivalent to equation (40) of Gebbie and Thomas (1968). Furthermore, inspection of equation (6) shows that the integral will decrease with increasing r_ν for *any* monotonically increasing temperature law, so $L_l > L_c$ is necessary for emission in any such situation.

For a given wavelength, which fixes y_0, the transition from conditions for absorption to emission will occur at the value of L_c for which $dF_\nu/dr_\nu = 0$. This condition is most easily satisfied for a weak line or line wing where $r_\nu \ll 1$. Equation (11) was integrated numerically, using an accurate representation of $q(\tau_c)$, for enough values of L_c and y_0 to map the locus of $dF_\nu/dr_\nu = 0$ for the case $L_l = 1$. This locus is the upper solid curve in

Figure 1. Emission cannot occur by the Schuster mechanism in a gray atmosphere with a value of L_c above this line. If $L_l < 1$, then the curve will be pushed to smaller values of L_c.

Let us consider the case in which inequality (10) is satisfied for $r_\nu \ll 1$. If we have a strong line, r_ν will vary from $r_\nu \ll 1$ in the wings to $r_\nu \gg 1$ at the line center. As r_ν increases, dF_ν/dr_ν will decrease and may become negative, so that the emission will be self-reversed. For the purpose of this discussion, let us say that a strong line is in emission

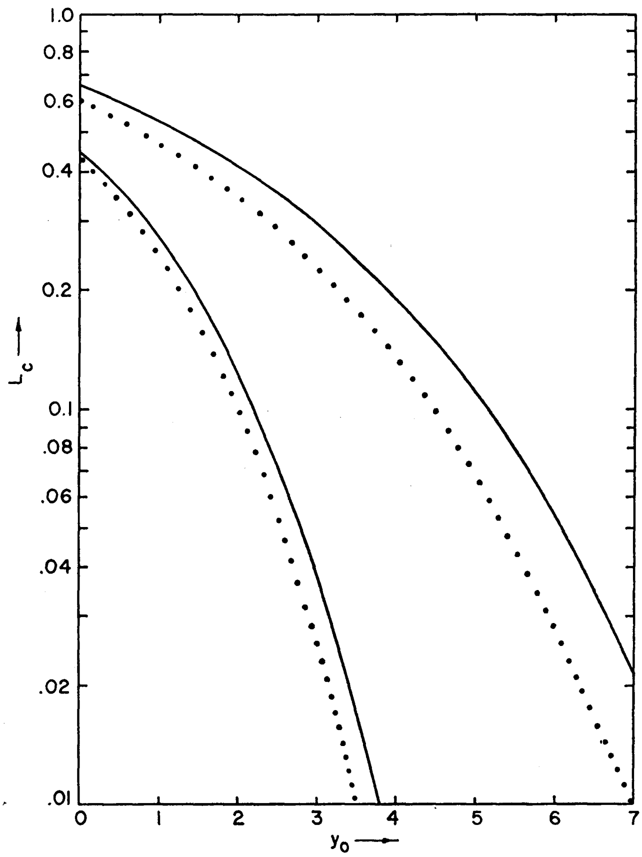

FIG. 1.—Maximum value of L_c for which emission can occur in a gray atmosphere for weak LTE lines (*upper solid curve*) and strong LTE lines (*lower solid curve*) as a function of $y_0 = hc/\lambda kT_0$. Dotted lines are approximations explained in text.

if the flux at the center is greater than the continuum level, regardless of the sign of dF_ν/dr_ν at the line center. Now in the limit $r_\nu \to \infty$, $F_\nu(0)$ is just

$$F_l(0) = \frac{4}{\sqrt{3}} \Lambda_l \frac{2h\nu^3}{c^2} [\exp(y_0) - 1]^{-1}, \qquad (12)$$

where $\Lambda_l = L_l^{1/2}/(1 + L_l^{1/2})$. The continuum flux is given by

$$F_c(0) = \frac{4}{\sqrt{3}} \Lambda_c \int_0^\infty B_\nu(\zeta) e^{-\zeta} d\zeta, \qquad (13)$$

where $\zeta = (3L_c)^{1/2}\tau_c$ and Λ_c is defined analogously to Λ_l. We will thus say that a strong line is in emission if $F_l(0) > F_c(0)$. The locus of $F_l(0) = F_c(0)$ was determined for $L_l = 1$ by numerical integration of equation (13) and is shown as the lower solid curve in Figure 1.

To understand the behavior of these criteria for emission, we consider a simple approximation: we evaluate equation (13) by assuming that the integral has the value $B_\nu(\zeta = 1)$. If we further take $q(\tau) \cong 1/\sqrt{3}$, the condition for $F_l(0) > F_c(0)$ is

$$\Lambda_c[\exp(y_0) - 1] < \Lambda_l[\exp(y_0\Lambda_c^{1/4}) - 1] . \qquad (14)$$

The solution of the equation corresponding to inequality (14) for $L_l = 1$ is shown by the lower dashed curve in Figure 1. The agreement with the solid curve is rather good. Applying the same approximation to equation (11) leads to the requirement that the quantity in braces must vanish at $\zeta = 1$ in order that $dF_\nu/dr_\nu = 0$. If we also assume that $dq/d\tau \cong 0$, then the condition for the onset of emission in weak lines becomes

$$\frac{y_{0c}\Lambda^{1/4}}{1 - \exp(-y_0\Lambda_c^{1/4})} = 4\left(\frac{L_l - L_c}{L_l + L_c}\right) . \qquad (15)$$

The solution of this equation for $L_l = 1$ is the upper dashed curve in Figure 1. While the values of L_c given by equation (15) are too low, the variation with y_0 is reproduced well. Alternatively, we could obtain expressions of greater accuracy by following Gebbie and Thomas (1968) and using a linear approximation to the Planck function, taking care, however, that this representation reflects the behavior of $B_\nu(\tau)$ to a depth of $\tau_\nu \approx (3L_\nu)^{-1/2}$.

We observe that, in principle, there is no limit to the wavelength at which an emission line can occur in an atmosphere of a given boundary temperature, provided we make L_c small enough—in the limit of large y_0, the solution of equation (15) becomes $L_c = (3.92/y_0)^8$. But to go to short wavelengths requires L_c to be too small to be of any practical interest.

We can best illustrate how the amplitude of emission varies with r_ν by a few examples, shown in Figure 2. We plot $F_\nu(0)/B_\nu(0)$ as a function of r_ν for specific values of L_c and $y_e = hc/\lambda kT_{\rm eff}$ ($T_{\rm eff}$ is the effective temperature; $y_0 = 1.233\, y_e$). These curves were obtained by numerical integration of equation (6).

The conditions necessary for the appearance of emission lines remain extreme, even in the most favorable case of the LTE line source function. Consider a line frequency of $\sim$4000 Å in an atmosphere with an effective temperature of 30000° K. We then have $y_e = 1.2$ as in Figure 2, *c*. If κ is due to the continuous absorption of H and He and if σ is due to electron scattering, then $L_c \leq 0.15$ requires that $n_e \leq 1.2 \times 10^{13}$ cm^{-3} (on the assumption of LTE). Furthermore, the assumption of constant L_c implies that this value (and the corresponding low density) must persist to optical depths in the continuum greater than $\sim(3L_c)^{-1/2}$, in this case $\tau_c \gtrsim 1.5$. But at this density, a layer of unit optical depth has a thickness of 1.5 $R_\odot$, an appreciable fraction of the radius of an O-type star. Such an extended envelope is quite different, for example, from the structure of an O-type atmosphere in hydrostatic equilibrium under an effective gravity of 10^3–10^4 cm sec^{-2}. In the latter case, L_c can be very small for $\tau_c \ll 1$, but because of the increasing density this situation does not persist to appreciable optical depths in the continuum.

III. THE COMPLETE-REDISTRIBUTION SOURCE FUNCTION

The preceding section detailed the conditions for emission in the most favorable case, that of the LTE line source function. If a line transition occurs between two levels maintained in LTE by collisions, the criteria in Figure 1 will be applicable. Such conditions could perhaps occur for a very weak subordinate line, but it will be particularly difficult to obtain the necessary collision rates at the low densities associated with small values of L_c. Such a weak line would imply a correspondingly weak emission feature. It is also possible for populations close to LTE to be maintained by radiative transitions other than those of the line itself, but each specific case would have to be examined in detail. We might also apply the $L_l = 1$ criteria to emission associated with discontinuities in the continuous absorption, provided that such absorption is close to LTE at the levels where the continuum is formed.

We now turn to strong or moderately strong lines. It is well known that the appropriate values of L_l are much less than unity. The coherent scattering source function given by equation (1) is not strictly applicable, however, since the line-broadening mechanisms in a stellar atmosphere lead to a sitaution close to complete redistribution. The simplest realistic case is the source function for the "two-level atom":

$$S_l = (1 - L_l) \int_0^\infty J_\nu \phi_\nu d\nu + L_l B_\nu . \qquad (16)$$

If we consider the center of a strong line ($r_{\nu_0} \gg 1$), the total source function will be essentially the line source function. Now the most favorable case for emission will occur for small values of y_0 where the gradient of $B_\nu(\tau_c)$ is small. For a sufficiently strong line we can then consider the Planck function to be constant at the surface value $B_\nu(0)$ over

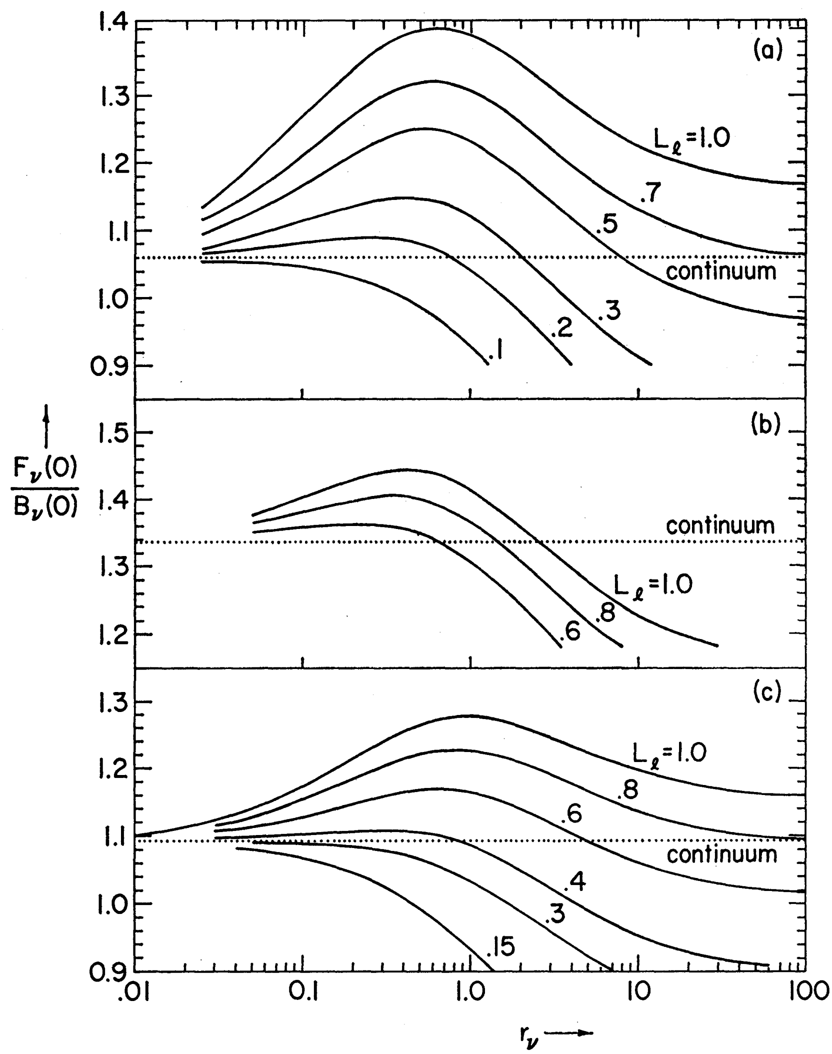

FIG. 2.—Ratios of line flux to the surface value of the Planck function, plotted against the ratio of line extinction to continuous extinction. The cases shown are: (*a*) $y_e = 2.0$, $L_c = 0.05$; (*b*) $y_e = 2$, $L_c = 0.15$; (*c*) $y_e = 1.2$, $L_c = 0.15$. Dotted line is $F_c(0)/B_\nu(0)$.

the region which affects the formation of the line (down to $\tau_c \sim (r_{\nu_0} L_l)^{-1}$ if ϕ_ν is a Doppler profile). The problem is then that of a line formed in an isothermal atmosphere, which has been investigated in detail (see Jefferies 1968). Both the complete-redistribution source function and the coherent-scattering source function have the same surface value, $S_l(0) = (L_l)^{1/2} B_\nu(0)$, but the former relaxes less quickly to the Planck function $B_\nu(0)$ than the latter. It follows that the flux at the center of a line formed by complete redistribution will be less than or equal to the expression for coherent scattering given by equation (12). So we must at least satisfy inequality (14) for emission. For example, with a typical value of $L_l = 10^{-4}$, at $y_0 = 1$ we require $L_c < 1.4 \times 10^{-6}$! Even the requirement that $L_c < L_l$ will rule out the Schuster mechanism as a realistic possibility. Thus, near the line center, the redistribution will only tend to make emission more difficult, which is the conclusion reached by Gebbie and Thomas.

The situation is different, however, in the *wings* of a strong line, which may be formed deep enough that the (noncoherent) line source function has relaxed to $S_l \cong B_\nu$. At least, this would seem to occur in normal stellar atmospheres where one can use the LTE source function to calculate line profiles which fit the observations away from the line center. In this case, the appropriate criteria for emission would be those for $L_l = 1$ presented above. Indeed, this may be the most likely situation in which the Schuster mechanism could contribute to emission, if it is of any real importance at all.

We may conclude that, although we have relaxed the conditions for the Schuster mechanism somewhat, it can still only operate in an abnormally extended, tenuous atmosphere where scattering dominates the continuous opacity to large optical depths. Even under such conditions, the mechanism would only be able to produce emission in line wings, or perhaps very weak lines; we cannot expect strong lines as a whole to appear in emission.

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