1. The Coupled Escape Probability Method in Spherical Symmetry

1.1. Absorption Probability Along a Specific Line-of-Sight

We consider a line with a Doppler profile, so that the (normalized) line profile function for absorption is

$$\phi(x) \;=\; \frac{1}{\sqrt{\pi}}\, e^{-x^2} \qquad \text{where} \qquad x = \frac{\nu - \nu_0}{\Delta\nu_D} \tag{1}$$

where $\Delta\nu_D$ is the Doppler width of the line. Then, with the assumption of complete redistribution, the distribution in frequency of the radiation emitted – by scattering or by thermal processes – is given by the same profile $\phi(x)$. The optical depth at frequency x is given by $\tau\phi(x)$, where τ is called the mean optical depth in the line. (Note that the line center optical depth $\tau(x = 0)$ is $\tau/\sqrt{\pi}$.) Thus the probability that radiation will be emitted at frequency x and travel optical depth τ without absorption is just $\phi(x)\; e^{-\tau\phi(x)}$. So we define the function

$$\eta(\tau) \;=\; \int_{-\infty}^{\infty} \phi(x)\; e^{-\tau\phi(x)}\; dx \tag{2}$$

Then, along a particular line-of-sight, the fraction of radiation intercepted between optical depth τ_1 and optical depth τ_2 will be $\eta(\tau_1) - \eta(\tau_2)$. This $\eta(\tau)$ is in some sense analogous to the $\alpha(\tau)$ of Elitzur and Ramos (2005) (ER05). Note that $\eta(\tau)$ is a smooth function which can be tabulated and easily interpolated for any τ. For small values of τ , a power-series expansion is useful.(1)

1.2. The Line Coupling Matrix for Spherical Shells

Consider a series of spheres of radius R_i for $i = 1, 2, ..., (N + 1)$, which bound N nested spherical shells. Consider a point at radius $R_i < r_i < R_{i+1}$ in the i^{th} shell. Let a ray from this point r_i which makes an angle θ with the radial direction (and define $\mu = \cos\theta$) ultimately cross the boundaries of shell j at points $\tau(\mu, R_j)$ and $\tau(\mu, R_{j+1})$. (For some μ the line may miss shells $j < i$. For other μs the line may cut the same shell twice. A line may also cut R_{j+1} twice, but not R_j.). The τ's must be calculated by summing up the segments $\kappa_k\; \Delta r(\mu, R_k, R_{k+1})$ through all the intervening shells. Here, $\Delta r(\mu, R_k, R_{k+1})$ represents the distance through shell k from r_i along the direction μ. Then the quantity $m_{ij}(\mu) = \eta[\tau(\mu, R_j)] - \eta[\tau(\mu, R_{j+1})]$ is the chance that radiation traveling in direction μ will be intercepted in shell j. If we then integrate over all angles, we obtain

$$m_{ij}(r_i) \;=\; \frac{1}{2} \int_{-1}^{1} [\eta(\tau(\mu, R_j)) - \eta(\tau(\mu, R_{j+1}))]\;\, d\mu \quad , \tag{3}$$

the probability that radiation leaving point r_i in shell i will be intercepted by shell j. The value of m_{ij} will vary with the position of r_i within the shell. Thus we must also integrate r_i over the volume of the shell, $dV_i = 4\pi r_i^2 dr_i$, for $R_i < r_i < R_{i+1}$, to obtain

$$M_{ij} \;=\; \frac{3}{R_{i+1}^3 - R_i^3} \int_{R_i}^{R_{i+1}} m_{ij}(r_i)\; r_i^2\; dr_i \tag{4}$$

and we call the array of M_{ij} the coupling matrix. Note that the value M_{ii} is the probability that the radiation is re-absorbed in the same shell from which it was emitted. We have written J code to compute this matrix given a set of shell radii $R_1, ..., R_{N+1}$ and shell opacities $\kappa_1, ...\kappa_N$.

1.3. The Line Source Function for the Two-Level Atom

Consider the line radiation emitted from a spherical shell j with volume V_j. This will be just $4\pi\mathcal{J}_j\; V_j$, where $\mathcal{J}$ is the emission coefficient. Now the source function is just $S = \mathcal{J}/\kappa$, so the radiation emitted from the shell is $4\pi\kappa_j\; S_j\; V_j$. Now the ji element of our coupling matrix M_{ji} is the probability that radiation emitted by shell j will be intercepted by shell i, so the radiation emitted by j and scattered in i is $4\pi\kappa_j\; S_j\; V_j\; M_{ji}$.

On the other hand, in terms of the mean intensity $\bar{J}_i$, the radiation scattered in shell i must be $4\pi\bar{J}_i\; \kappa_i\; V_i$. If we denote by $\bar{J}_{ij}$ the the mean intensity in shell i which originates in shell j, then we can write the radiation emitted in j and scattered in i as $4\pi\bar{J}_{ij}\; \kappa_i\; V_i$. Equating this to the expression in the previous paragraph and summing over all emitting shells j we have

$$\kappa_i\; \bar{J}_i\; V_i \;=\; \sum_{j=1}^{N} \kappa_j\; V_j\; M_{ji}\; S_j \tag{5}$$

which leads to our expression for the mean intensity in shell i:

$$\bar{J}_i \;=\; \sum_{j=1}^{N} \left(\frac{\kappa_j}{\kappa_i}\right) \left(\frac{V_j}{V_i}\right)\; M_{ji}\; S_j \tag{6}$$

Now the line source function for the two-level atom is given by

$$S_i \;=\; (1-\epsilon_i)\; \bar{J}_i \;+\; \epsilon_i\; B_i \tag{7}$$

so the equation for the source function S_i becomes

$$S_i \;-\; (1-\epsilon_i) \sum_{j=1}^{N} \left(\frac{\kappa_j}{\kappa_i}\right) \left(\frac{V_j}{V_i}\right) M_{ij}\; S_j \;=\; \epsilon_i B_i \tag{8}$$

or, with I representing the identity matrix, we have the matrix equation

$$\left[I_{ij} \;-\; (1-\epsilon_i) \left(\frac{\kappa_j}{\kappa_i}\right) \left(\frac{V_j}{V_i}\right) M_{ij} \right] \;\times\; [S_i] \;=\; [\epsilon_i B_i] \tag{9}$$

1.4. Multi-Level Atoms: The Net Radiative Bracket

The CEP treatment developed by ER05 makes use of the "net radiative bracket" of Athay and Skumanich (ER05, eq. 6):

$$p(\tau) \;=\; 1 - \frac{\bar{J(\tau)}}{S(\tau)} \tag{10}$$

From our expression for the mean intensity given above, we thus have

$$p_i \;=\; 1 \;-\; \sum_{j=1}^{N} \left(\frac{\kappa_j}{\kappa_i}\right) \left(\frac{V_j}{V_i}\right) M_{ji} \; \frac{S_j}{S_i} \tag{11}$$

This can be inserted into the code we developed for the plane-parallel problems to provide solutions to the corresponding problems in spherical symmetry.

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(1) If τ is small, a useful expression for $\eta(\tau)$ can be obtained by expanding the exponential in equation (2):

$$\eta(\tau) \;=\; \int_{-\infty}^{\infty} \phi(x) \left\{ 1 \;-\; \tau\,\phi(x) \;+\; \frac{\tau^2}{2}\,\phi^2(x) \;-\; \cdots \right\} dx \;=\; \sum_{n=0}^{\infty} \frac{(-\tau)^n}{n!} \int_{-\infty}^{\infty} \phi^{n+1}(x)\, dx$$

and since

$$\phi^k \;=\; \pi^{-k/2}\, e^{-kx^2} \qquad \text{and} \qquad \int_{-\infty}^{\infty} e^{-kx^2}\, dx \;=\; \sqrt{\frac{\pi}{k}}$$

we have

$$\eta(\tau) \;=\; \sum_{n=0}^{\infty} \frac{(-1)^n}{\pi^{n/2}\; n!\; \sqrt{n+1}}\; \tau^n$$

Explicitly, the first few terms are

$$\eta(\tau) \;\simeq\; 1 \;-\; 0.39894228\,\tau \;+\; 0.09188815\,\tau^2 \;-\; 0.01496559\,\tau^3 \;+\; 0.00188801\,\tau^4 \;-\; \cdots$$