THE ASTROPHYSICAL JOURNAL, 439:828–845, 1995 February 1

CRITICAL RADIATION FLUXES AND LUMINOSITIES OF BLACK HOLES AND RELATIVISTIC STARS

FREDERICK K. LAMB[1,2,3] AND M. COLEMAN MILLER[1,2,4]
Received 1994 May 9; accepted 1994 August 5

ABSTRACT

The critical luminosity at which the outward force of radiation balances the inward force of gravity plays an important role in many astrophysical systems. We present expressions for the radiation force on particles with arbitrary cross sections and analyze the radiation field produced by radiating matter, such as a disk, ring, boundary layer, or stellar surface, that rotates slowly around a slowly rotating gravitating mass. We then use these results to investigate the critical radiation flux and, where possible, the critical luminosity of such a system in general relativity.

We demonstrate that if the radiation source is axisymmetric and emission is back–front symmetric with respect to the local direction of motion of the radiating matter, as seen in the comoving frame, then the radial component of the radiation flux and the diagonal components of the radiation stress-energy tensor outside the source are the same, to first order in the rotation rates, as they would be if the radiation source and gravitating mass were not rotating. If the opacity is independent of frequency and direction, the critical flux for matter at the surface of a star or in orbit around a star or black hole is the same, at least to first order, as it would be if the matter, radiation source, and gravitating mass were static. In this case the critical flux *measured at the radiation source* is also the same to first order as it would be if the matter, source, and mass were static. We argue that the critical radiation flux for matter at rest in the locally nonrotating frame is often satisfactory as an astrophysical benchmark flux and show that if this benchmark is adopted, many of the complications potentially introduced by rotation of the radiation source and the gravitating mass are avoided. If instead the opacity is frequency- or direction-dependent, the critical flux generally depends on the angular size and spectrum of the source and is affected by rotation of the source and mass and orbital motion of the matter to first order.

We show that if the radiation field in the absence of rotation would be spherically symmetric and the opacity is independent of frequency and direction, one can define a critical luminosity for the system that is independent of the spectrum and angular size of the radiation source and is unaffected by rotation of the source and mass and orbital motion of the matter, to first order. Finally, we analyze the conditions under which the maximum possible luminosity of a star or black hole powered by steady spherically symmetric radial accretion is the same in general relativity as in the Newtonian limit.

Subject headings: accretion: accretion disks — black hole physics — radiative transfer — relativity — stars: neutron

1. INTRODUCTION

The critical luminosity at which the outward force of radiation balances the inward force of gravity was originally introduced in the context of stellar structure (Eddington 1926). This concept has since been found useful in understanding not only the luminosities of massive stars (Zel'dovich & Novikov 1971), but also a wide variety of other astrophysical phenomena (Zook & Berg 1975; Bowers & Deeming 1984; Katz 1987). The critical luminosity plays an especially important role in X-ray bursts (see Lewin, van Paradijs, & Taam 1993), accretion by neutron stars and black holes (see Shapiro & Teukolsky 1983; Lamb 1991a; Mészáros 1992), quasi-periodic brightness oscillations in the most luminous X-ray sources (Lamb 1989, 1991b; Fortner, Lamb, & Miller 1989, 1994; van der Klis 1994a, b), and active galactic nuclei (see Begelman, Blandford, & Rees 1984; Rees 1984; Abramowicz, Ellis, & Lanza 1990; Svensson 1990).

There are several reasons for investigating further the critical luminosities of black holes and relativistic stars. First, previous analyses have assumed that the cross section is independent of the frequency and direction of the incident radiation. However, in many models of the inner regions of active galactic nuclei, a substantial fraction of photons have energies $h\nu \gtrsim m_e c^2$ (see Guilbert, Fabian, & Rees 1983; Svensson 1987, 1990; Krolik 1988; Done & Fabian 1989; Björnsson & Svensson 1992; Zdziarski 1992), in which case the electron scattering cross section depends on the frequency ν of the radiation. Cross sections near magnetic neutron stars generally depend on angle as well as frequency (see Bussard, Alexander, & Mészáros 1986; Daugherty & Harding 1986; Wang, Wasserman, & Salpeter 1988; Graziani 1993).

Second, the standard expression for the critical luminosity assumes that both the gravitating mass and the radiating matter are static. Yet most black holes and relativistic stars and especially the radiating matter around them are expected to be rotating (see, e.g., Bardeen 1970a; Ghosh & Lamb 1979; Czerny, Czerny, & Grindlay 1986; Blandford & Rees 1992). It

[1] Department of Physics, University of Illinois at Urbana-Champaign, 1110 West Green Street, Urbana, IL 61801.
[2] Nordic Institute for Theoretical Physics, Blegdamsvej 17, DK-2100 København Ø, Denmark.
[3] University of Illinois at Urbana-Champaign, Department of Astronomy; f-lamb@uiuc.edu.
[4] University of Chicago, Department of Astronomy and Astrophysics, 5640 South Ellis Avenue, Chicago, IL 60637; miller@gamma.uchicago.edu.

is therefore important to know how accurate the standard expression is for such systems.

A third reason for reconsidering the critical luminosities of relativistic stars is that several authors (Nemiroff, Becker, & Wood 1993; see also Walker 1992) have argued recently that previous derivations of the critical luminosity for Thomson scattering are incorrect, and that the critical luminosity depends on the angular size of the radiation source; according to Nemiroff et al., the critical luminosity in some circumstances is 50% greater than the previously accepted value.

Here we first present expressions for the radiation force on particles with arbitrary cross sections. We then investigate the radiation field produced by a radiation source, such as a disk, ring, boundary layer, or stellar surface, that rotates slowly around a slowly rotating gravitating mass. We use the relativistic equation of motion to generalize the usual definition of the critical radial force outside a nonrotating gravitating mass to the case of a slowly rotating mass. We then use this definition to define and compute critical radiation fluxes and, where possible, critical luminosities for matter at or outside the gravitating mass. For conciseness we sometimes discuss relativistic stars even though the results apply equally well to black holes. Where our results apply only to stars, we indicate this explicitly.

By "slowly rotating" we mean that the angular momentum J of the gravitating mass is much less than GM^2/c and that the appropriately averaged azimuthal velocity $\langle v_\phi \rangle$ of the radiating matter as measured in the locally nonrotating frame or LNRF (see Bardeen 1970b; Bardeen, Press, & Teukolsky 1972) is much less than c. In practice, the requirement $j \equiv cJ/GM^2 \ll 1$ is not very restrictive. For example, all neutron stars with known periods are slowly rotating in this sense ($j \approx 0.2$ for a uniform density spherical star with a mass of 1.4 $M_\odot$, a radius of 10 km, and a spin period of 1.6 ms). Black holes in binary systems are also expected to be slowly rotating according to this criterion because the accretion phase for black holes with high-mass companions is too short for the hole to accrete much angular momentum, while the total mass and hence angular momentum that can be accreted from a low-mass companion is too small (Miller & Lamb 1995). The angular momenta of massive black holes in active galactic nuclei are more uncertain, but some may have $j \ll 1$, depending on how they are formed and fueled (Blandford 1990).

The requirement $v \equiv \langle v_\phi \rangle / c \ll 1$ is also usually satisfied if the black hole or neutron star is slowly rotating. Radiating matter on the surface of a slowly rotating neutron star satisfies $v \lesssim 0.2$, while the azimuthal velocity of matter in a viscous boundary layer at the surface of such a star may range from $\lesssim 0.2$ to $\lesssim 0.5c$, so that $v \lesssim 0.35$. In neutron star or black hole systems with luminosities greater than a few percent of the critical luminosity, radiating matter experiences substantial azimuthal radiation drag (see Miller & Lamb 1993, 1995), so that v is likely to be $\lesssim 0.3$. For systems with near-critical luminosities, the orbital velocity of radiating matter in near-Keplerian orbits is small, due to the large radial component of the radiation force, and the radiation drag force is even stronger (see Fortner et al. 1989; Lamb 1989, 1991b).

The paper is organized as follows. In § 2 we derive expressions for radiation forces near relativistic objects. In § 3 we compute the radiation stress-energy tensor produced by nonrotating and rotating radiation sources at the surface of or outside nonrotating and slowly rotating relativistic stars and black holes. In § 4, we use the expressions for the radiation force derived in § 2 and the results for the radial component of the radiation flux obtained in § 3 to compute critical fluxes and luminosities for static and moving matter in systems composed of slowly rotating black holes or relativistic stars and slowly rotating radiation sources. Our results are summarized in § 5.

2. RADIATION FORCE

In this section we consider radiation forces near relativistic objects. We first discuss the force on a particle at rest in a radiation field, paying special attention to the effects of frequency- and angle-dependence of the differential cross section. Next, we discuss methods of treating the force on a particle in motion. We conclude this section with a brief discussion of how the force on an element of gas can be calculated from knowledge of the forces on the particles that make up the gas.

2.1. *Force on a Particle at Rest*

Consider first the radiation force on a particle that is momentarily at rest, or equivalently, the force produced by the radiation field seen in an inertial frame momentarily comoving with the particle.

2.2.1. *General Expressions*

A particle in a radiation field may absorb, emit, or scatter radiation. For simplicity, we assume that the radiation is unpolarized and that the interaction of the particle with the radiation field is linear, i.e., that processes such as induced scattering can be neglected. Then the force on a particle at position x^γ produced by scattering of the radiation in an infinitesimal range $d\nu$ around frequency ν propagating in an infinitesimal solid angle $d\Omega$ about the direction $\mathbf{n}$ in the rest frame of the particle is

$$\frac{1}{c} I(\mathbf{n}, \nu; x^\mu) \int_0^\infty d\nu' \int_{4\pi} d\Omega' \frac{d\sigma}{d\Omega'} (\mathbf{n}, \nu; \mathbf{n}', \nu') \left(\mathbf{n} - \frac{\nu'}{\nu} \mathbf{n}' \right) d\nu \, d\Omega \,, \tag{1}$$

where $I(\mathbf{n}, \nu; x^\gamma)$ is the specific intensity of the incident radiation propagating in direction $\mathbf{n}$, $d\sigma/d\Omega'$ is the differential scattering cross section, ν' is the frequency of the radiation scattered in direction $\mathbf{n}'$, and the relation between ν and ν' imposed by momentum conservation is assumed to be included in the expression for the differential cross section. Here and below Greek indices run over the space and time components of vectors and tensors, Roman indices run only over the space components, boldface indicates a 3-vector, and we use the spacelike sign convention for the metric (signature $-+++$). If the angle and frequency distribution of the outgoing radiation is such that its momentum in the rest frame of the particle is zero, the term $-\nu'\mathbf{n}'/\nu$ in the final parentheses of equation (1) vanishes when the integration over Ω' is performed.[5] This is the case for electron scattering in the Thomson limit, but not for Compton scattering. The force produced by absorption is $(1/c)I(\mathbf{n}, \nu; x^\gamma)\sigma(\mathbf{n}, \nu)\mathbf{n}\, d\nu\, d\Omega$, where $\sigma(\mathbf{n}, \nu)$ is the absorption cross section.

The force on the particle produced by its interaction with the full radiation field is obtained by integrating expression (1)

[5] The outgoing radiation need not be isotropic; if $\nu' = \nu$, it is sufficient that the radiation pattern be unchanged under inversion of the coordinates.

over **n** and ν. Hence the k-component of the full radiation force, in a local orthonormal coordinate system comoving with the particle, may be written

$$f^{\hat{k}}_{\rm rad} = \frac{1}{c}\int_0^\infty d\nu \int_{4\pi} d\Omega\, I(\mathbf{n}, \nu; x^\gamma)\,\sigma^{\hat{k}}(\mathbf{n}, \nu)\,, \tag{2}$$

where

$$\sigma^{\hat{k}}(\mathbf{n}, \nu) = \int_0^\infty d\nu' \int_{4\pi} d\Omega' \frac{d\sigma}{d\Omega'}(\mathbf{n}, \nu; \mathbf{n}', \nu')\left(\mathbf{n} - \frac{\nu'}{\nu}\mathbf{n}'\right)\cdot \mathbf{k} \tag{3}$$

is the cross section for momentum transfer in the k-direction.[6] Here and below we use hats on indices to indicate the components of vectors and tensors expressed in a local orthonormal tetrad consisting of an orthonormal set of 4-vectors, three spacelike and one timelike. If the distribution of scattered radiation is such that its net momentum is parallel (or antiparallel) to the direction of the incident radiation, as in Compton scattering, the momentum transferred to the particle will be parallel to **n** and hence $\sigma^{\hat{k}} \propto n^{\hat{k}}$. If the particle absorbs radiation, the force is given by expression (2) with $\sigma^{\hat{k}}(\mathbf{n}, \nu) = \sigma(\mathbf{n}, \nu)\,\mathbf{n}\cdot\mathbf{k}$, where $\sigma(\mathbf{n}, \nu)$ is the cross section for absorption. The radiation force cannot be written in the simple form (2) if the interaction of the particle with the radiation is nonlinear.

Frequency- and angle-independent cross section.—Suppose that σ is independent of both the *frequency* and the *direction* of the incident radiation. Then the cross section can be removed from the integral on the right side of equation (2), and the expression for the k-component of the radiation force at x^μ reduces to

$$f^{\hat{k}}_{\rm rad} = \frac{1}{c}\,\sigma F(n^{\hat{k}}; x^\mu)\,, \tag{4}$$

where $F(n^{\hat{k}})$ is the radiation flux in the k-direction; in terms of the radiation stress tensor $T^{\hat{\alpha}\hat{\beta}}$ in the local orthonormal coordinate system,

$$F(n^{\hat{k}}; x^\gamma) = cT^{\hat{t}\hat{k}} = c\int_0^\infty d\nu \int_{4\pi} d\Omega\, I(\mathbf{n}, \nu; x^\gamma)\, n^{\hat{k}}\,. \tag{5}$$

Thus, *in this special case the radiation force is proportional to, and in the same direction as the radiation flux at the position of the particle.*

Frequency- and/or angle-dependent cross section.—If σ depends on the frequency or the direction of the incident radiation, equations (2) and (5) show that the k-component of the radiation force can still be written

$$f^{\hat{k}}_{\rm rad} = \frac{1}{c}\,\langle\sigma(n^{\hat{k}})\rangle F(n^{\hat{k}})\,, \tag{6}$$

where

$$\langle\sigma(n^{\hat{k}})\rangle = \frac{\int_0^\infty d\nu \int_{4\pi} d\Omega\, I(\mathbf{n}, \nu; x^\gamma)\,\sigma^{\hat{k}}(\mathbf{n}, \nu)}{\int_0^\infty d\nu \int_{4\pi} d\Omega\, I(\mathbf{n}, \nu; x^\gamma)\, n^{\hat{k}}} \tag{7}$$

is the appropriately frequency- and angle-averaged cross section at the position of the particle. Expressions (6) and (7) show that the magnitude of the force on the particle generally is not proportional to the radiation flux, nor is the force on the particle generally in the same direction as the radiation flux, since $\langle\sigma(n^{\hat{k}})\rangle$ will usually depend on **k**. Whether expression (6) is useful depends on the situation.

[6] $\sigma^{\hat{k}}$ is also the cross section for attenuation of the radiation momentum and energy fluxes in the k-direction; it is the cross section for attenuation of the photon number flux only if $\nu' = \nu$.

2.1.2. *Examples*

Consider now, as specific examples, Compton scattering, absorption by an oriented flat particle (such as a dust grain), and scattering by a charged particle in a magnetic field. The first process often dominates near relativistic objects while the second and third processes illustrate the behavior of radiation forces when the cross section depends on angle.

Compton scattering.—The differential cross section for Compton scattering of unpolarized radiation is (see Berestetskiĭ, Lifshitz, & Pitaevskiĭ 1971, p. 297)

$$\frac{d\sigma}{d\Omega}(\mathbf{n}, \nu; \mathbf{n}', \nu') = \frac{1}{2}\left(\frac{e^2}{mc^2}\right)^2\left(\frac{\nu'}{\nu}\right)^2\left(\frac{\nu}{\nu'} + \frac{\nu'}{\nu} - \sin^2\theta'\right), \tag{8}$$

where θ' is the angle between $\mathbf{n}'$ and **n**. Figure 1 shows that the resulting momentum transfer cross section $\sigma_p(\nu)$ is a steep function of the frequency ν of the incident radiation for $h\nu \gtrsim 0.1mc^2$; it is independent of the direction of the radiation. As noted above, this cross section is the same as that for attenuation of the radiation momentum and energy fluxes. Figure 1 also shows, for comparison, the cross section $\sigma_n(\nu)$ for attenuation of the photon number flux, the momentum transfer cross section $\sigma'_p(\nu)$ that would be obtained if the momentum impart-

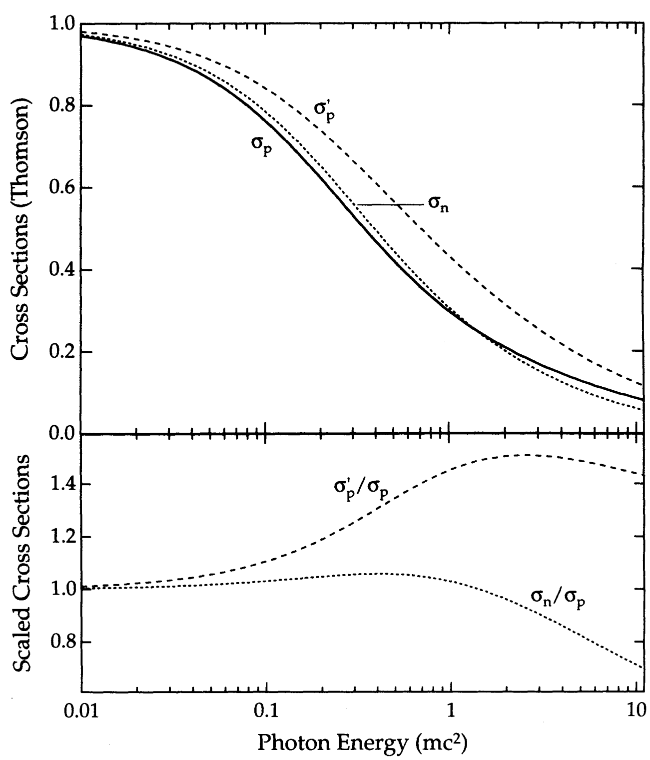

FIG. 1.—*Top:* Cross sections σ_p for momentum transfer (*solid line*) and σ_n for attenuation of the photon number flux (*dotted line*) by Compton scattering, in units of the Thomson cross section, as functions of the energy of the incident photon in units of mc^2. Also shown for comparison is the momentum transfer cross section σ'_p (*dashed line*) that would be obtained if the momentum imparted by the scattered photon were (incorrectly) neglected. *Bottom:* Cross section ratios σ_n/σ_p (*dotted line*) and σ'_p/σ_p (*dashed line*) as functions of frequency.

ed by the scattered radiation were (incorrectly) neglected, and the ratios $\sigma'_p(\nu)/\sigma_p(\nu)$ and $\sigma_n(\nu)/\sigma_p(\nu)$. All three cross sections are equal in the Thomson limit, but all three differ when $h\nu/mc^2$ cannot be neglected. Because the momentum transfer cross section depends steeply on the frequency of the incident radiation unless $h\nu < 0.1mc^2$, *the radiation force on the scattering particle is in the same direction as the radiation flux with a magnitude proportional to the flux only in the Thomson limit.*

Absorption by a flat particle.—Consider now the radiation force on a flat, infinitesimally thick, perfectly absorbing small but macroscopic (dimensions much greater than the wavelength λ of the radiation) particle of area σ_0. The cross section presented by such a particle to radiation propagating in a given direction depends on its orientation but not on the frequency of the radiation.

In order to obtain explicit expressions for $f^{\hat{k}}_{\text{rad}}$, we introduce a local orthonormal coordinate system with polar angle $\tilde{a}$ and azimuthal angle $\tilde{b}$, centered on the particle and oriented so that the polar axis ($\tilde{a} = 0$) points in the $-k$-direction. We will also need to refer to a direction perpendicular to **k**, which we choose to be the direction $\tilde{a} = \pi/2$, $\tilde{b} = \pi/2$; for consistency with the coordinate system used in § 3, we call this the ϕ-direction. Then $n^{\hat{k}} = \cos\tilde{a}$ and $n^{\hat{\phi}} = \sin\tilde{a}\sin\tilde{b}$. Because the cross section is independent of frequency, we can work with the frequency-integrated intensity $I(\mathbf{n}, x^\gamma)$.

Suppose first that the particle is oriented with its faces perpendicular to **k**. The force in the k-direction on the face with outward normal in the $-k$-direction is due to radiation propagating in the k-direction, which imparts momentum in that direction to the particle; the force on the surface facing in the k-direction is due to radiation propagating in the $-k$-direction, which imparts momentum in that direction to the particle. Integrating the radiation stress over the two surfaces of the particle, one finds that the net force on the particle in the k-direction is

$$f^{\hat{k}}_{\text{rad}} = \frac{1}{c}\int_{4\pi} d\Omega\, I(\mathbf{n};\, x^\gamma)\,\sigma_0\,|\cos\tilde{a}|\cos\tilde{a}\,, \tag{9}$$

where $d\Omega = \sin\tilde{a}\,d\tilde{a}\,d\tilde{b}$. Comparison with equation (2) shows that in this case, the momentum transfer cross section is $\sigma^{\hat{k}}(\mathbf{n}) = \sigma_0\,|\cos\tilde{a}|\cos\tilde{a}$. If instead the particle is oriented so that its normal is in the ϕ-direction, a similar analysis shows that

$$f^{\hat{k}}_{\text{rad}} = \frac{1}{c}\int_{4\pi} d\Omega\, I(\mathbf{n};\, x^\gamma)\,\sigma_0\,\sin\tilde{a}\,|\sin\tilde{b}|\cos\tilde{a}\,. \tag{10}$$

Comparison with equation (2) shows that for this orientation, the momentum transfer cross section is $\sigma^{\hat{k}}(\mathbf{n}) = \sigma_0 \sin\tilde{a}\,|\sin\tilde{b}|\cos\tilde{a}$. The cross section for attenuation of the photon energy and number fluxes are equal to the momentum transfer cross section.

It is instructive to compare the forces (9) and (10) with the momentum carried by the radiation field, which is described by the spatial components

$$T^{\hat{\imath}\hat{\jmath}} = \int_{4\pi} d\Omega\, I(\mathbf{n})\, n^{\hat{\imath}} n^{\hat{\jmath}} \tag{11}$$

of the radiation stress-energy tensor. The diagonal spatial component

$$T^{\hat{k}\hat{k}} = \int_{4\pi} d\Omega\, I(\mathbf{n})\,\cos\tilde{a}\,\cos\tilde{a} \tag{12}$$

describes the flux of k-momentum flowing in the k-direction and is sometimes said to represent the radiation "pressure" in the k-direction. This might be misunderstood to imply that the k-component of the radiation force on a flat particle oriented with its faces perpendicular to **k** is proportional to $T^{\hat{k}\hat{k}}$. Comparing expression (9) with expression (12) shows that *the k-component of the radiation force on a flat particle oriented with its faces perpendicular to* **k** *generally is not proportional to the flux of k-momentum in the k-direction.* The k-component of the radiation force is proportional to $T^{\hat{k}\hat{k}}$ if radiation falls only on the face of the particle with outward normal in the $-k$-direction; however, in general radiation also falls on the face with outward normal in the k-direction.

The flux of k-momentum flowing in the ϕ-direction is

$$T^{\hat{k}\hat{\phi}} = \int_{4\pi} d\Omega\, I(\mathbf{n})\,\sin\tilde{a}\,\sin\tilde{b}\,\cos\tilde{a}\,. \tag{13}$$

Comparing expression (10) with expression (13) shows that *the k-component of the radiation force on a flat particle oriented with its faces perpendicular to the ϕ-direction generally is not proportional to the flux of k-momentum in the ϕ-direction*, either. The reason is that the k-momenta of rays coming from the two sets of directions $0 < b < \pi$ and $\pi < b < 2\pi$ contribute with the same sign to the k-momentum imparted to the particle but contribute with opposite signs to $T^{\hat{k}\hat{\phi}}$. A similar result holds if the particle is oriented with its faces in the k, ϕ-plane.

Scattering by a charged particle in a magnetic field.—When radiation is scattered by a charged particle in a magnetic field, the cross section for momentum transfer generally depends on the frequency and direction of the incident radiation (see, e.g., Bussard et al. 1986; Daugherty & Harding 1986; Wang et al. 1988; Graziani 1993). In this case the radiation force is not proportional to the radiation flux, or to any other component of the radiation stress tensor.

2.1.3. *Discussion*

As mentioned in the Introduction, the radiation force on a particle such as an electron is often loosely attributed to radiation "pressure." The preceding analysis shows that the magnitude of the radiation force on a particle with a momentum transfer cross section that depends on frequency or angle generally is not proportional to the diagonal "pressure" components of the radiation stress-energy tensor nor to *any* component of the radiation stress-energy tensor, including the radiation flux. Moreover, the force on such a particle generally is not in the same *direction* as the local radiation flux. Only if the cross section is independent of both angle and frequency, as in Thomson scattering, is the radiation force in the same direction as the radiation flux in the rest frame of the particle, with a magnitude that is proportional to the flux.

The important distinction between the radiation pressure and the radiation force on a particle is illustrated by the case of an isotropic radiation field, which has nonzero pressure (an evacuated, reflecting balloon would be collapsed by the radiation pressure) but no net flux; it therefore exerts no force on a particle with a cross section that is angle-independent. However, even an isotropic radiation field may exert a force on a particle, if the cross section for momentum transfer is angle-dependent (a flat particle with one face that absorbs radiation and one face that reflects it would be accelerated by the radiation field).

In their recent discussion of the critical luminosity of a static, spherically symmetric relativistic star, Nemiroff et al. (1993)

write (incorrectly) the radial component of the radiation force on an electron-proton pair as (in our notation).[7]

$$f^{\hat{r}} = \text{“}P_{\rm rad}\text{”}\sigma_{\rm T} \equiv \frac{1}{c}\int_0^\infty d\nu \int_{4\pi} d\Omega\, I(\mathbf{n}, \nu)\,\sigma_{\rm T}\,|\cos\tilde{\alpha}|\cos\tilde{\alpha}\,, \quad (14)$$

where $\sigma_{\rm T}$ is the Thomson cross section and the axis of the local orthonormal coordinate system has been chosen in the $-r$-direction. Here the quantity "$P_{\rm rad}$" is neither the radiation "pressure" (eq. [12]) nor the radiation flux (eq. [5]). Comparison of equation (14) with equation (9) shows that "$P_{\rm rad}$"$\sigma_{\rm T}$ would be the radial component of the radiation force on an infinitesimally thin particle of area $\sigma_{\rm T}$ oriented with its faces perpendicular to the local radial direction. However, a Thomson-scattering electron does not interact with radiation like such a surface, since the Thomson cross section is angle-independent. Hence, the factor $|\cos\tilde{\alpha}|$ in the integrand of equation (14) is inappropriate (see eqs. [4] and [5]). The conclusion of Nemiroff et al. that the standard expression for the critical luminosity is incorrect is due entirely to this error in computing the radiation force, as we show in § 4.

2.2. *Force on a Particle in Motion*

Up to now we have discussed computation of the radiation force in a local orthonormal coordinate system momentarily comoving with the particle. However, in astrophysical problems it is often more convenient to specify the radiation intensity and other quantities in a global coordinate system related to some object other than the particle, such as a star, and to analyze the motion of the particle in the global coordinate system. If this approach is followed, the intensity must first be transformed from the global coordinate system to a local orthonormal system, in order to compute the force on the particle, and the force on the particle must then be transformed to the global coordinate system (see, e.g., Shapiro & Teukolsky 1983, chap. 5).

In this work we are interested in relativistic stars. If the exterior geometry is Schwarzschild and the particle is at rest in the Schwarzschild coordinate system, the required transformations are relatively simple. If instead the exterior geometry is that produced by a rotating, axisymmetric star (see, e.g., Friedman, Ipser, & Parker 1986; Cook, Shapiro, & Teukolsky 1992, 1994), the required transformations are more complicated but are still relatively simple when the particle is at rest in the LNRF, where the effects of frame-dragging are minimized (see Bardeen 1970b; Bardeen et al. 1972). If the particle is not at rest in the static frame or, in the case of a rotating star, in the LNRF, the required transformations are considerably more complex.

The problem of computing the motion of the particle in the global coordinate system is greatly simplified if the cross section for momentum transfer is independent of the direction and frequency of the radiation, since in this case the force on the particle in the comoving frame is proportional to the radiation flux in the comoving frame, as discussed in § 2.2.1. One can therefore use the fact that the momentum flux density *in the rest frame of the particle* may be written as $T^{\alpha\beta}u_\beta$, where u^β is the particle's four-velocity and all quantities are expressed in the global coordinate system (see, e.g., Misner, Thorne, & Wheeler 1973, p. 131), to write the radiation force on the particle, in the global coordinate system, as (see Abramowicz et al. 1990)

$$f^\alpha_{\rm rad} = \frac{\sigma}{c}\, h^\alpha{}_\beta\, T^{\beta\nu}u_\nu\,, \quad (15)$$

where $h^\alpha{}_\beta = -\delta^\alpha{}_\beta - (u^\alpha u_\beta)/c^2$ projects the components perpendicular to u^β and we have used the Einstein summation convention.

2.3. *Force on a Element of Gas*

So far we have discussed the radiation force on a particle with momentum transfer cross section $\sigma^k(\mathbf{n}, \nu)$ However, in most astrophysical problems one is interested in the radiation force on a small volume of gas composed of several species of particles rather than the force on a single particle.

Under fairly general conditions, the radiation force on a small volume of gas is equal to the sum of the forces on the individual particles. Some of the necessary conditions are that the photon mean free path be much greater than the interparticle spacing, that the particles be independently and randomly distributed in space, and that the particles be coupled together by, for example, collisions. If, in addition, the velocity distribution of the particles is such that the velocity-averaged momentum transfer cross section is equal to the momentum transfer cross section for a particle at rest in the local center-of-momentum frame of the gas, then the k-component of the force per unit mass, in the comoving frame of the gas is

$$f^k_{\rm rad} = \frac{1}{c}\,\kappa(n^k)F(n^k)\,, \quad (16)$$

where

$$\kappa(n^k) = \frac{\sum_i n_i \langle\bar{\sigma}_i(n^k)\rangle}{\sum_i n_i m_i}\,. \quad (17)$$

Here n_i is the number density, m_i is the mass, $\langle\bar{\sigma}_i(n^k)\rangle$ is the intensity-weighted cross section (7) for particles of species i, and the sums run over all species; the bar indicates that the cross section is to be averaged over the orientations of the particles. Expression (17) generally is not valid if particle thermal velocities are comparable to the speed of light, so that terms of order $(v_{\rm th}/c)^2$ in the Lorentz transformations to the center-of-momentum frame are important, or if there are large electrical currents, so that the streaming velocities of particles in the center-of-momentum frame of the gas are important.

In the preceding sections we showed that the expression for the radiation force acting on an individual particle is much simpler if the momentum transfer cross section is frequency- and angle-independent. The opacity (eq. [17]) of a gas is generally direction-independent even if the cross sections of the individual particles are not, unless the particles are oriented or the angle dependence of their cross sections is related to some special direction within the system, such as that defined by a magnetic field. The average over particle orientations does not reduce any frequency dependence of the opacity caused by frequency dependence of the momentum transfer cross section.

If the mean free path of all photons is small compared to the distance over which the relevant properties of the gas change,

[7] Nemiroff et al. write θ where we write $\tilde{\alpha}$. Puzzlingly, they refer to θ as "the angle between the detector normal and the line of sight." From the context it appears that θ is actually the angle between the direction $\mathbf{n}$ of an arbitrary ray incident on the particle and the radial direction. Nemiroff et al. also write I_ν throughout their derivation without indicating any integration over frequency. Presumably this integration was intended, and we have therefore included it in eq. (14).

the gas acts as a "container" for the radiation. If in addition the opacity is independent of angle and frequency, the force on the gas may be related to the *gradient* of the radiation pressure, since in this case the radiation energy flux is given by $F(n^{\hat{k}}) = -D(\partial\epsilon/\partial x_{\hat{k}}) = -3D(\partial P_{\rm rad}/\partial x_{\hat{k}})$, where $D = c/(3\kappa\rho)$ is the diffusion coefficient in terms of the mass density ρ of the gas, ϵ is the radiation energy density, and $P_{\rm rad}$ is the (isotropic) radiation pressure ($T^{\hat{k}\hat{k}} = P_{\rm rad}\,\delta^{\hat{k}\hat{k}}$). Inserting this expression for $F(n^{\hat{k}})$ in equation (4), one finds that the k-component of the force per unit mass is $-(1/\rho)(\partial P_{\rm rad}/\partial x_{\hat{k}})$. Obviously, this expression is less general than equation (4), since equation (4) is valid even if the photon mean free path is large.

3. RADIAL COMPONENT OF THE RADIATION FLUX

In this section we compute the radial component of the radiation flux near a nonrotating or slowly rotating radiation source located at or outside the surface of a relativistic star or the horizon of a black hole that is nonrotating or slowly rotating. For our purposes a star or black hole is slowly rotating if $j \equiv J/M^2 \ll 1$, where J and M are the angular momentum and gravitational mass of the star or black hole; here and below we set $c = G = 1$. As discussed in § 1, all neutron stars with known spin rates are slowly rotating in this sense. Black holes in binary systems are expected to be slowly rotating and some black holes in active galactic nuclei may have $j \ll 1$.

In general, the geometry of spacetime outside a rotating relativistic star depends on the structure of the star (see, e.g., Friedman et al. 1986; Cook et al. 1992, 1994) and differs significantly from the geometry outside the horizon of a rotating black hole (see Miller & Lamb 1992; Cook, Shapiro, & Teukolsky 1994). If, however, the star is axisymmetric and slowly rotating, then the spacetime geometry outside it is unique to first order in j, depends only on its mass and angular momentum, and is the same as the geometry outside a black hole with the same mass and angular momentum (Hartle & Thorne 1968; the effect on the geometry of the deviation from sphericity caused by rotation of a star is of second and higher order in j). Thus, spacetime around *any* slowly rotating star or black hole can be described by this two-parameter family of geometries.

We begin by giving the transformations between coordinate systems that we will need later in this section and in § 4. Next, we compute the angular size and the radiation flux as a function of radius for a static, spherically symmetric radiation source around a nonrotating star or black hole. Finally, we consider the radial component of the radiation flux produced by a slowly rotating radiation source in the rotation equator of a slowly rotating star or black hole.

3.1. *Coordinate Transformations*

Below we compute the radial flux in the locally nonrotating frame or LNRF (Bardeen 1970b; Bardeen et al. 1972), which reduces to the static frame as $j \to 0$. However, we also wish to consider various elements of the radiation stress-energy tensor in frames with azimuthal velocities different from that of the LNRF, as well as in global Schwarzschild (for $j = 0$) or first-order (in j) Boyer-Lindquist (1967) coordinates (for $j \neq 0$). We therefore give here the transformations needed to go from one to another of these coordinate systems.

If the radiation stress-energy tensor in a given tetrad is $T^{\hat{\mu}\hat{\nu}}$, the stress-energy tensor in a tetrad moving with velocity β in the ϕ-direction relative to the original tetrad is

$$T^{\hat{\mu}'\hat{\nu}'} = L^{\hat{\mu}'}{}_{\hat{\mu}} L^{\hat{\nu}'}{}_{\hat{\nu}} T^{\hat{\mu}\hat{\nu}} , \tag{18}$$

where

$$L^{\hat{r}'}{}_{\hat{r}} = L^{\hat{\theta}'}{}_{\hat{\theta}} = 1 , \quad L^{\hat{t}'}{}_{\hat{t}} = L^{\hat{\phi}'}{}_{\hat{\phi}} = \gamma \equiv (1 - \beta^2)^{-1/2} ,$$
$$L^{\hat{t}'}{}_{\hat{\phi}} = L^{\hat{\phi}'}{}_{\hat{t}} = -\gamma\beta , \tag{19}$$

and all other elements of the Lorentz transformation tensor $L^{\hat{\mu}'}{}_{\hat{\mu}}$ are zero.

The world line of a particle that is at rest in the LNRF is, in Boyer-Lindquist coordinates, $r = $ const., $\theta = $ const., and $\phi = \omega t + $ const., where $\omega = -g_{\phi t}/g_{\phi\phi} \approx 2jM^2/r^3$ is the angular velocity of the LNRF as seen from infinity. The world line of a particle with an arbitrary azimuthal velocity may be written

$$r = \text{const.} , \quad \theta = \text{const.} , \quad \text{and} \quad \phi = \lambda\omega t + \text{const.} , \tag{20}$$

where the dimensionless parameter λ describes the angular velocity of the particle relative to the static frame, in units of the angular velocity of the LNRF as seen from infinity. If $\lambda = 0$, the particle is at rest as seen by an observer at infinity whereas if $\lambda = 1$, the particle is at rest in the LNRF.

We denote the transformation from the tetrad comoving with a particle to the first-order Boyer-Lindquist coordinate system by $e^{\mu}{}_{\hat{\nu}}$; for a particle with the world line (20), the nonzero elements of $e^{\mu}{}_{\hat{\nu}}$ are, to first order in λ and j,

$$e^{t}{}_{\hat{t}} = \left(1 - \frac{2M}{r}\right)^{-1/2} , \quad e^{r}{}_{\hat{r}} = \left(1 - \frac{2M}{r}\right)^{1/2} ,$$
$$e^{\theta}{}_{\hat{\theta}} = \frac{1}{r} , \quad e^{\phi}{}_{\hat{\phi}} = \frac{1}{r \sin\theta} ,$$
$$e^{\phi}{}_{\hat{t}} = \frac{2jM^2\lambda}{r^3(1 - 2M/r)^{1/2}} , \quad e^{t}{}_{\hat{\phi}} = -\frac{2jM^2(1 - \lambda)}{r^3(1 - 2M/r)^{1/2}} . \tag{21}$$

Note that the diagonal elements of the transformation (21) and its inverse are the same (to first order in j) as for a nonrotating gravitating mass.

If the gravitating mass is not rotating, $j = 0$ and the transformation (21) and its inverse show that the radial component of the radiation flux in the Schwarzschild coordinate system is equal to the radial component in the comoving tetrad to first order in λ, i.e.,

$$F(n^r) \equiv T^{tr} = e^{t}{}_{\hat{\mu}} e^{r}{}_{\hat{\nu}} T^{\hat{\mu}\hat{\nu}} = T^{\hat{t}\hat{r}} \equiv F(n^{\hat{r}}) \tag{22}$$

and vice versa, regardless of the structure of $T^{\hat{\mu}\hat{\nu}}$. However, the *nonradial* components of the radiation flux in the two coordinate systems generally are *not* equal to this order.

If the gravitating mass is rotating, $j \neq 0$ and the radial flux $F(n^r)$ in the Boyer-Lindquist coordinate system is in general equal to the radial flux $F(n^{\hat{r}})$ in the comoving tetrad to $\mathcal{O}(j)$ if and only if the particle is at rest in the LNRF ($\lambda = 1$). Conversely, $F(n^{\hat{r}})$ is in general equal to $F(n^r)$ to $\mathcal{O}(j)$ if and only if the particle is at rest in the LNRF. The two radial fluxes generally are not equal to $\mathcal{O}(j^2)$, even if the particle is at rest in the LNRF.

3.2. *Nonrotating Star or Black Hole and Radiation Source*

Consider now the radial component of the radiation flux at or outside a static, spherically symmetric radiation source of radius $R_{\rm em}$ centered on but outside a static, spherically symmetric distribution of gravitating matter, such as a nonrotating star or black hole, with mass M. We are interested in situations in which the radiation field at and outside $R_{\rm em}$ is stationary and spherically symmetric about $r = 0$. If the gravitating mass is a star, the radiation source could be its surface, a static (pressure-supported) spherically symmetric distribution of matter outside the surface, or a spherically symmetric distribution of inflowing matter. If the gravitating mass is a black hole, the radiation source could be a spherically symmetric distribution of inflowing matter outside the event horizon. We define the radius of the source as the radius outside which the luminosity does not change. In general, matter may be present outside the star or black hole and the radiation source and may interact with the radiation there. We assume that the gravitational mass of any such matter is negligible compared to the mass of the star or black hole, so that it does not affect the geometry of spacetime outside the star or black hole.

Suppose first that there is no matter outside the source. For simplicity we assume that the source emits isotropically (i.e., with a specific intensity that is independent of direction) as well as uniformly. We present here expressions for the frequency-integrated specific intensity and radiation flux outside the source, since we will need these expressions later in this section and the next. For this purpose, we introduce a unit spacelike vector $\boldsymbol{n}$ with local tetrad components $n^{\hat{k}}$. We specify spacelike directions by their polar angle $\tilde{a}$ and azimuthal angle $\tilde{b}$ in a coordinate system oriented so that the polar axis ($\tilde{a} = 0$) points radially *inward*, the direction ($\tilde{a} = \pi/2$, $\tilde{b} = 0$) is parallel to the polar axis of the Schwarzschild (or below, the Boyer-Lindquist) coordinate system, and ($\tilde{a} = \pi/2$, $\tilde{b} = \pi/2$) points in the local ϕ-direction. Then

$$n^{\hat{r}} = \cos \tilde{a} \, , \quad n^{\hat{\theta}} = \sin \tilde{a} \cos \tilde{b} \, , \quad n^{\hat{\phi}} = \sin \tilde{a} \sin \tilde{b} \, , \tag{23}$$

and the radial tetrad component of the radiation flux at r is (compare eq. [5])

$$T^{\hat{t}\hat{r}} = \int_0^{2\pi} d\tilde{b} \int_0^{\pi} d\tilde{a} \sin \tilde{a} \, I(\tilde{a}, \tilde{b}; r) \cos \tilde{a} \, , \tag{24}$$

where $I(\tilde{a}, \tilde{b}; r)$ is the frequency-integrated specific intensity at r in the direction $(\tilde{a}, \tilde{b})$.

The frequency-integrated specific intensity I observed along a given light ray is proportional to the fourth power of the redshift from the point of emission to the point of reception (one power comes from the change in photon frequency, one from time dilation, and two from gravitational defocussing, which causes the solid angle of a ray bundle to vary as the square of the redshift). In the Schwarzschild geometry the redshift from any point on the source to any point x^k outside the source is a function only of the radius $R_{\rm em}$ of the source and the radius r of the point outside. Thus, the specific intensity at x^k is (see Abramowicz et al. 1990)

$$I_0(r) \equiv I(R_{\rm em}) \left(\frac{1 - 2M/R_{\rm em}}{1 - 2M/r} \right)^2 \tag{25}$$

for all rays that reach x^k from the source and zero for all other rays. Here $I(R_{\rm em})$ is the frequency-integrated specific intensity at the source and the subscript 0 indicates that neither the gravitating mass nor the radiation source are rotating. The specific intensity distribution at r is therefore

$$I_0(\mathbf{n}; r) = I_0(r) \begin{cases} 1 \, , & \tilde{a} \le \alpha_0 \, , \\ 0 \, , & \tilde{a} > \alpha_0 \, , \end{cases} \tag{26}$$

where α_0 is the half-angle of the limb of the source as seen at r; it is given implicitly by (Abramowicz et al. 1990)

$$\sin \alpha_0 = \frac{R_{\rm em}}{r} \left(\frac{1 - 2M/r}{1 - 2M/R_{\rm em}} \right)^{1/2} \times \begin{cases} 1 \, , & R_{\rm em} \ge 3M \, , \\ \dfrac{3\sqrt{3}}{2} \dfrac{2M}{R_{\rm em}} \left(1 - \dfrac{2M}{R_{\rm em}} \right)^{1/2} \, , & R_{\rm em} \le 3M \, . \end{cases} \tag{27}$$

The variation of α_0 with the radius of the observer differs for sources larger and smaller than $3M$ because some of the photons emitted from a source of radius $R_{\rm em} < 3M$ are reabsorbed, whereas if $R_{\rm em} > 3M$, all photons emitted from the source escape to infinity. For $R_{\rm em} \le 3M$, α_0 does not depend on $R_{\rm em}$.

Because the radiation field is spherically symmetric, the radiation flux at a given point x^k outside the source is in the radial direction and depends only on the Schwarzschild radial coordinate r. Substituting the specific intensity distribution (26) into equation (24), one finds that the radial flux measured in a static tetrad at $r \ge R_{\rm em}$ is

$$F(n^{\hat{r}}; r) = \left(\frac{1 - 2M/R_{\rm em}}{1 - 2M/r} \right) F(n^{\hat{r}}; R_{\rm em}) \, , \tag{28}$$

where $F(n^{\hat{r}}; R_{\rm em})$ is the radial flux measured at the source. One factor of the redshift comes from the reduction of the photon frequency and one from the reduction of the photon arrival rate. The Lorentz transformation (eq. [19]) shows that the radial flux measured in any tetrad moving in the azimuthal direction with velocity β is, to first order in β, equal to the flux measured in the static tetrad and is therefore also given by equation (28) to this order. The transformation (eq. [21]) shows that the flux measured at r in the Schwarzschild coordinate system is equal to the flux measured at r in the static tetrad, i.e., $F(n^r; r) = F(n^{\hat{r}}; r)$. Hence

$$F(n^r; r) = \left(\frac{1 - 2M/R_{\rm em}}{1 - 2M/r} \right) \frac{L(R_{\rm em})}{4\pi R_{\rm em}^2} = \frac{1}{(1 - 2M/r)} \frac{L(\infty)}{4\pi r^2} \, , \quad r \ge R_{\rm em} \, , \tag{29}$$

where $L(R_{\rm em}) \equiv 4\pi R_{\rm em}^2 F(n^r; R_{\rm em})$ is the luminosity measured at $R_{\rm em}$ and $L(\infty)$ is the luminosity measured at radial infinity. Although in deriving equations (28) and (29) we assumed that matter in the source emits isotropically, this is not necessary; spherical symmetry of the radiation field is sufficient. When F is expressed in terms of $L(\infty)$, it is independent of the angular size and radius of the source.

Suppose now that matter is present outside the source and interacts with the radiation there. So long as the matter does not change the luminosity and the radiation field remains spherically symmetric at r, the radiation flux at r is still given by equation (29) and is therefore independent of the optical depth of any matter between the source and r, the angular size of the radiation source seen at r (which depends on the optical depth if the matter scatters radiation), and the optical depth of any matter between r and infinity. The radial flux is given by equation (29) whether r is at the surface of the radiation source, in a region of negligible optical depth outside the source, or in a spherically symmetric, optically thick cloud surrounding the source.

If the radiation field produced by the source is asymmetric, then in general the radial component of the radiation flux at a given point x^k outside the source depends not only on the luminosity and the radius corresponding to x^k, but also on the direction. If scattering or absorbing matter is present, the flux at a given point will generally depend on the distribution of the matter and determination of the flux will require an analysis of radiation transport in the neighborhood of the source. However, if the matter is distributed spherically symmetrically and has a large optical depth, the radiation field at r may be nearly spherically symmetric even if emission from the source is asymmetric. In the case of accretion onto a black hole, equation (29) applies provided the accretion flow is spherically symmetric and does not change the luminosity outside R_{em}.

3.3. *Rotating Star or Black Hole and Radiation Source*

Next consider the radiation field produced by a system composed of a rotating, stationary axisymmetric radiation source, such as a disk or ring, rotating around a (possibly differentially) rotating, stationary axisymmetric distribution of gravitating matter with mass M. For conciseness we will refer to the gravitating mass as a star; however, our results apply equally well if the mass is a black hole.

We assume that the star is rotating about its symmetry axis. As before, we parameterize the (appropriately averaged) azimuthal velocity of the radiating matter by v, as measured in the LNRF, and the angular momentum of the star by j and assume that the gravitational mass of any matter outside the star is so small that it does not affect the geometry of spacetime there. The spacetime outside such a star is stationary, axisymmetric, and asymptotically flat, and the metric may therefore be written in the standard form (eq. [2.4]) of Bardeen et al. (1972).

The radiation source may be the star's surface, an axisymmetric portion of it, or some other axisymmetric distribution of matter. We assume that the source is rotating about its symmetry axis, and that this axis is co-aligned with the rotation axis of the star. The radiation source may rotate in the same or the opposite sense as the star and need not be symmetric with respect to the plane defined by the rotation equator of the star; the specific intensity radiated by matter in the source may depend on direction. Nevertheless, the radiation field produced by such a system depends only on the radial coordinate r and colatitude θ defined in the standard form of the metric.

Several important aspects of the radiation field produced by a system of this kind are illustrated by explicit computation of the radial component of the radiation flux for two simple examples, namely, emission by a uniformly radiating, rotating thin ring of radius R_{em} in the rotation equator of the star and emission by a similar half-ring.

To compute the radiation flux, we use first-order Boyer-Lindquist coordinates (r, θ, ϕ) centered on the star and introduce a tetrad at the measurement point, which we take to be at rest in the LNRF at an arbitrary radius r outside the surface of the star. We specify directions in the tetrad at r using the same spacelike unit vector $\boldsymbol{n}$ and the same polar and azimuthal angles $\tilde{a}$ and $\tilde{b}$ introduced in § 3.2. The radial component $T^{\hat{t}\hat{r}}$ of the radiation flux in the tetrad may be computed from the specific intensity distribution using equation (24).

Consider first the radiation field of the full ring. In order to obtain a simple expression for the intensity distribution measured by an observer at rest in the LNRF outside the star, we make some simplifying assumptions about the emission from the ring. Introducing a unit vector $\boldsymbol{n}'$ that is spacelike as seen by a local observer riding on the ring, we assume for simplicity that the specific intensity of the radiation is independent of $\boldsymbol{n}'$, i.e., that the ring radiates isotropically. Our earlier assumption that the ring radiates uniformly means that the specific intensity seen by a locally comoving observer is also independent of ϕ.

To determine the specific intensity distribution at the measurement point, we again use the completely general result that along a given path the frequency-integrated specific intensity I varies as ν^4, regardless of whether the frequency ν of the ray is changed by gravitational redshifts, Doppler shifts, frame-dragging, cosmological redshifts, or any other similar effect or combination of effects. For the full ring, we find (Miller & Lamb 1995) that the specific intensity measured by an observer who is at rest in the LNRF outside the star is, to first order in v and j,

$$I(\boldsymbol{n}; r) \approx I_0(\boldsymbol{n}; r)\left\{1 + \frac{4r \sin \tilde{a} \sin \tilde{b}}{(1 - 2M/r)^{1/2}} \times \left[\frac{v}{R_{em}}\left(1 - \frac{2M}{R_{em}}\right)^{1/2} + 2j\left(\frac{M^2}{R_{em}^3} - \frac{M^2}{r^3}\right)\right]\right\}, \quad (30)$$

where $I_0(\boldsymbol{n}; r)$ is equal to $I_0(r)$ (see eq. [25]) in the directions $\boldsymbol{n}$ of the ray paths that intersect the ring and zero otherwise. This expression is valid for an observer at any radius $r > R$ and any colatitude θ. The term $2j(M^2/R_{em}^3 - M^2/r^3)$ in the bracket on the right side of equation (30) represents the frequency shift due to frame dragging while the term $(v/R_{em})(1 - 2M/R_{em})^{1/2}$ describes the Doppler shift caused by the rotation of the ring. For a radiation source of more general but still axisymmetric form, such as a uniformly rotating and uniformly emitting spherical source of luminous rings at various latitudes, the factor v on the right in equation (30) must be replaced by the appropriate average velocity.

The Doppler effect on the intensity distribution is much larger than the effect of frame dragging, because the angular velocity Ω_{em} of the matter in the ring is much greater than the angular velocity ω of the LNRF at R_{em}. To see this, note that the velocity v of the radiating matter as measured by an observer in the LNRF in the rotation equator at R_{em} is $(\Omega_{em} - \omega) R_{em}(1 - 2M/R_{em})^{-1/2}$. Therefore, the first term in the square brackets on the right side of equation (30), which is the Doppler effect, is equal to $\Omega_{em} - \omega$. At radial infinity, where the LNRF coincides with the static frame, the second term in the square brackets on the right side of equation (30), which is the frame-dragging effect, is equal to ω, to first order. Hence the ratio of the Doppler and frame-dragging effects seen at radial infinity is $(\Omega_{em}/\omega) - 1$. Now $j = I\Omega/M^2 = \alpha R^2\Omega/M$, where I is the stellar moment of inertia and α is the square of the radius of gyration in units of the stellar radius, so $\Omega_{em}/\omega \approx \Omega_{em} R_{em}^3/2jM^2$; hence the ratio of the Doppler and frame-dragging effects is $(1/2\alpha)(R_{em}/R)^2(R_{em}/M)(\Omega_{em}/\Omega) - 1$. For realistic neutron stars, α is $\sim$0.3 (Friedman et al. 1986; Cook et al. 1994), so for $\Omega_{em} \geq \Omega$, $R_{em} \geq 4M$, and $r \gg R_{em}$ the ratio of effects is ≥ 5; for larger R_{em} or smaller r the ratio is even larger. This is a specific illustration of the general result that if the angular velocity of the radiation source is similar to or greater than the angular velocity of the gravitating mass, the Doppler effects on the intensity distribution are much larger than the effects of frame-dragging (see Friedman et al. 1986; Miller & Lamb 1995).

For an observer who is not in the rotation equator, the ring appears as a complicated curve in $\tilde{a}$ and $\tilde{b}$. However, for an equatorial observer at rest in the LNRF at any radius $r > R_{em}$, the ring is seen as a straight line in the rotation equator, extending from $\tilde{a} = \alpha_1(\pi/2)$ to $\tilde{a} = \alpha_1(3\pi/2)$, where $\alpha_1(\tilde{b})$ is given implicitly by

$$\sin \alpha_1(\tilde{b}) = \sin \alpha_0 \left[1 - 2j \sin \tilde{b} \frac{M^2}{R_{em}^2}\left(1 - \frac{R_{em}^3}{r^3}\right) \times \left(1 - \frac{2M}{R_{em}}\right)^{-1/2}\right] + \mathcal{O}(j^2) \quad (31)$$

and α_0 is given implicitly by equation (27). Note that although the angular position of the ring is shifted by an amount $\mathcal{O}(j)$, the angle subtended by the ring at any point in the rotation equator at $r > R_{em}$ is the same, to first order in $v = R_{em}\Omega_{em}$ and j, as if neither the gravitating mass and nor the emitting ring were rotating. For a source with a more general but still axisymmetric shape, such as the surface of a star, the same result holds; the source is shifted in position by an amount that is first order in j, but its angular area remains unchanged to this order (see Miller & Lamb 1995).

The radial component of the radiation flux from the full ring can be computed by substituting expression (30) for $I(\mathbf{n}; r)$ into equation (24) and performing the indicated integrals over $\tilde{a}$ and $\tilde{b}$. The terms proportional to $v \sin \tilde{b}$ and $j \sin \tilde{b}$ integrate to zero, showing that the lowest order corrections to the radial flux are second order in v and j. The transformation (eq. [21]) demonstrates that this is also true for the radial component of the radiation flux in the Boyer-Lindquist coordinate system.

To show that this result is not general, suppose now that only half the ring (the half defined by $\tilde{b} = -\pi/2$, say) emits radiation as before. It is intuitively clear that if the emitting half of the ring is the half in which matter is moving toward the observer, the flux will be increased; if instead the emitting half of the ring is the half in which matter is moving away from the observer, the flux will be decreased. Hence the sense of rotation matters and the change in the radial flux caused by the rotation of the source is $\mathcal{O}(v)$. This change is due to the Doppler effect. In general, there is also a change $\mathcal{O}(j)$ caused by the rotation of the gravitating mass. This is confirmed by substituting the specific intensity distribution (30) with appropriately modified angular limits into equation (24) and computing the radial flux. The terms proportional to $v \sin \tilde{b}$ and $j \sin \tilde{b}$ do not integrate to zero, so that the lowest order corrections are indeed $\mathcal{O}(v)$ and $\mathcal{O}(j)$.

The results for the ring and half-ring illustrate the fact that rotation of the radiation source and the gravitating mass generally produces changes $\mathcal{O}(v)$ and $\mathcal{O}(j)$, respectively, in the radiation field, with the effects caused by the motion of the matter in the source generally more important than the effects caused by frame-dragging. This was the case for both the ring and the half-ring. However, if the matter in the source emits symmetrically in the forward and backward directions and, like the full ring, is symmetric about its rotation axis, the radial flux measured by an observer at rest in the LNRF is unchanged to $\mathcal{O}(v)$ and $\mathcal{O}(j)$.

The result for the full ring is a specific example of a more general result, namely, that *the radial component of the radiation flux measured by an observer moving with an azimuthal velocity $\beta \ll 1$ relative to the LNRF is the same, to first order in j, v, and β, as it would be if the star, source, and observer were not rotating, provided that the radiation source is axisymmetric about its rotation axis and the emission from the matter in the source, as seen by an observer comoving with it, is unchanged by reflection through the local (r, θ)-plane.* A mathematical proof of this is given in Miller & Lamb (1995), but it can also be demonstrated using an argument based on the principle that a physical quantity can change to first order in the (signed) quantities j, v, and β if and only if its value depends on the senses of rotation of the star, source, and observer. The argument goes as follows.

Consider emission from a thin axisymmetric ring of matter rotating slowly around a slowly rotating star (see Fig. 2). Assume that the rotation axes of the ring and star are coaligned. The ring may be at or outside the surface of the star and need not be in the star's rotation equator. Suppose the colatitude of the ring relative to the rotation equator of the star is θ_{em}. Assume that the emission from any small element of the ring as seen by a comoving observer is unchanged by reflection through the local (r, θ)-plane. Then, as measured in the LNRF, the elements $T^{\hat{t}\hat{\phi}}$, $T^{\hat{r}\hat{\phi}}$, and $T^{\hat{\theta}\hat{\phi}}$ of the radiation stress-energy tensor are first-order in v and j; all other elements are in general of order unity (Miller & Lamb 1995). If the source is mirror-symmetric with respect to the rotation equator, the elements $T^{\hat{t}\hat{\theta}}$, $T^{\hat{r}\hat{\theta}}$, and $T^{\hat{\theta}\hat{\phi}}$ are zero.

Let F be the radial component of the radiation flux measured by an observer moving with azimuthal velocity $\beta_i \ll 1$ with respect to the frame that appears static at infinity, at some arbitrary colatitude θ_{obs} and some arbitrary spherical radius R_{obs} outside the star. This is the initial configuration C_1. If the observer's azimuthal velocity is changed by the small amount $\Delta\beta_i = -\beta_i$ needed to bring the observer to rest in the static frame, the Lorentz transformation (19) shows that the radial flux measured by the observer is still equal to F, to first order in β_i. This is configuration C_2.

Now consider the configuration C_3 produced by rotating configuration C_2 by π radians about the axis defined by the intersection of the rotation equator and the meridional plane containing the observer (see Fig. 2). In configuration C_3 the senses of rotation of the star and ring are opposite to those in C_1 and C_2; the beam originally seen by the observer is now on the other side of the star while the corresponding beam on the near side of the star points away from the observer. Next, reflect the star, ring, and observer through the rotation equator. In this rotated and mirror-imaged configuration, $\theta''_{obs} = \pi - \theta_{obs}$, $\theta''_{em} = \pi - \theta_{em}$, and the emission pattern is mirrored; the senses of rotation of the star and ring remain opposite to those in the initial configuration C_1, and the beam on the near side of the star still points away from the observer. However, if emission from the matter in the source is unchanged by reflection though the local (r, θ)-plane, the intensity of the beam pointing toward the observer is the same as that in the beam pointing away, and hence the radial flux measured by the observer in configuration C_4 is still F. Configuration C_4 is identical to configuration C_2, except that the senses of rotation of the star and ring are reversed.

If, finally, the observer's azimuthal velocity is changed by the small amount $\Delta\beta_f = \beta_i$ needed to give the observer the same azimuthal velocity as in the initial configuration C_1, the final configuration C_5 is identical to the initial configuration, except that the senses of rotation of the star and the ring are reversed. The Lorentz transformation (19) shows that the radial flux measured by the observer in the final configuration is equal to the flux F measured in the initial configuration, to first order.

Because changing the sense of rotation of the star or the ring does not change the radial flux, there can be no change in the radial flux to first order in j or v. Also, as we have seen, there is no change in the radial flux measured by the observer to first

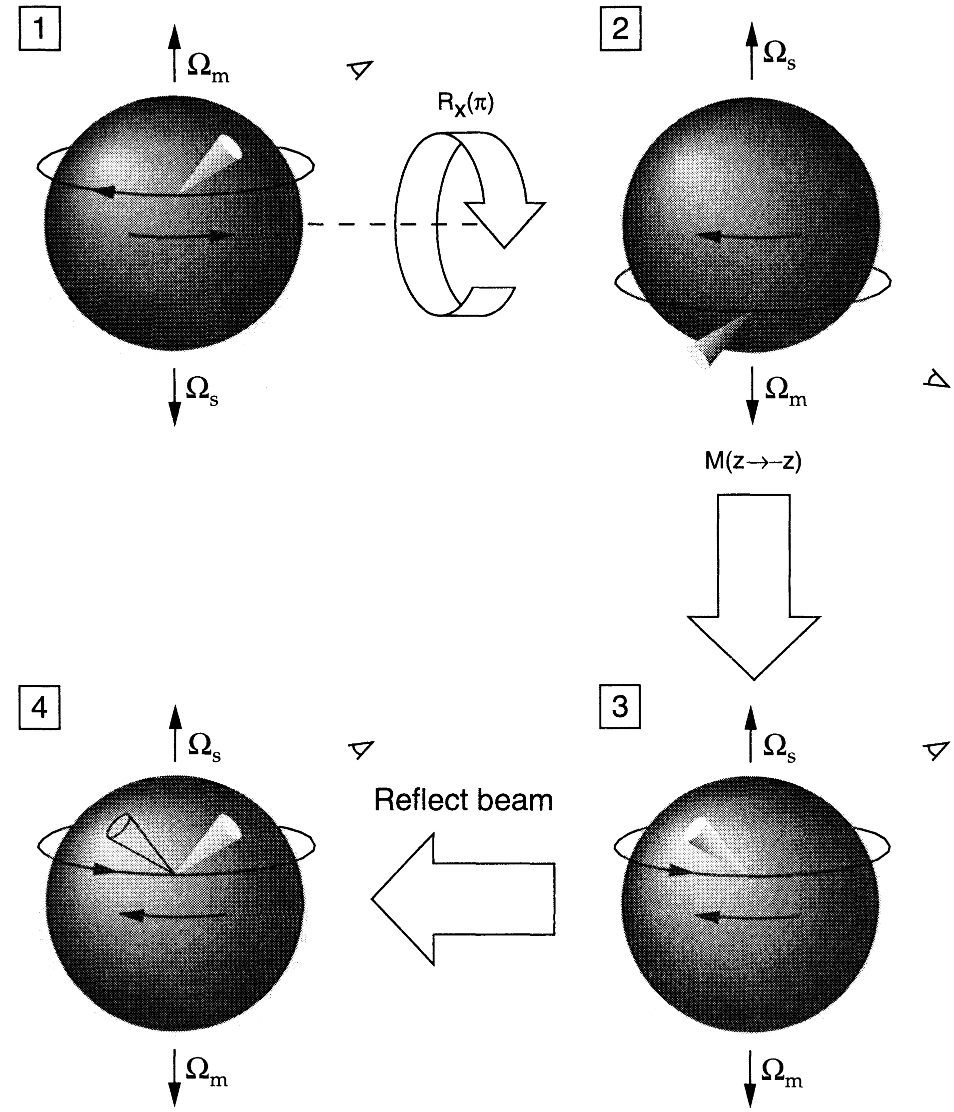

FIG. 2.—Sequence of operations described in the text which demonstrate that the radial component of the radiation flux and the diagonal components of the radiation stress-energy tensor do not depend on the rotation rates of the gravitating mass or the radiation source, to first order. [1] Configuration C_2 of a system composed of an axisymmetric ring source rotating slowly around a slowly rotating star or black hole and an observer. In this configuration the observer is at rest in the static frame, matter in the ring rotates clockwise about the vertical axis, and the gravitating mass rotates counterclockwise about the same axis. Also shown is one of the beams of radiation coming from the matter in the ring. The radial component of the radiation flux measured by the observer is F. [2] Configuration C_3 produced by rotating the initial configuration by π radians about the axis defined by the intersection of the rotation equator and the meridional plane containing the observer. The flux measured by the observer is again F. [3] Configuration produced by reflecting the rotated initial configuration through the rotation equator. The flux measured by the observer is still F. [4] Configuration C_4 produced by reflecting the radiation beam through the local r, θ-plane. Provided that emission from the matter in the ring is symmetric with respect to the local r, θ-plane, configuration C_4 is identical to configuration C_2, except that the sense of rotation of the star and the ring are reversed. The radial flux measured by the observer in configuration C_4 is the same as the flux F measured by the observer in configuration C_2.

order in the observer's azimuthal velocity β. Since any axisymmetric radiation source which emits radiation that is locally symmetric with respect to the (r, θ)-plane can be constructed of such rings, the radial flux measured by an observer rotating slowly around such a source is the same, to first order in j, v, and β, as if the mass, source, and observer were not rotating.

In addition to the radial component of the radiation flux, the θ-component of the flux as well as the off-diagonal element $T^{\hat{r}\hat{\theta}}$ and all the diagonal elements of the radiation stress-energy tensor measured by a slowly rotating observer are unaffected, to first order, by rotation of the radiation source and gravitating mass and by the motion of the observer. In contrast, the ϕ-component of the radiation flux and the off-diagonal elements $T^{\hat{r}\hat{\phi}}$ and $T^{\hat{\theta}\hat{\phi}}$ of the stress-energy tensor are all first-order in v, j, and β.

To summarize, if a radiation source in axisymmetric and rotating slowly about a slowly rotating mass and if emission from the matter in the source is backward-forward symmetric with respect to the local direction of motion of the matter, the radial component of the radiation flux measured in a slowly rotating tetrad is the same, to first order in the rotation rates of the source, star, and tetrad, as it would be if none of them were rotating. If these symmetries of the radiation source are absent, as in the case of the half-ring, the radial flux will in general depend on the rotation rates of the star, source, and tetrad to first order.

If a system without rotation would produce a spherically symmetric radiation field, then to first order, the radiation field produced by the system with rotation remains spherically symmetric and the radial flux measured in a slowly rotating tetrad is given by equation (29), i.e., by the same expression as in the absence of rotation. Even if the system with rotation produces an asymmetric radiation field, if there is a spherically symmetric and optically thick distribution of matter between the radiation source and the observer, the radiation field at the radius of the observer may be nearly spherically symmetric, so that the radial flux measured in a slowly rotating tetrad is again given by equation (29).

4. CRITICAL FLUXES AND LUMINOSITIES

In this section we use the expressions for the radiation force and the radial component of the radiation flux obtained in the previous two sections to compute critical radiation fluxes and luminosities for a slowly rotating radiation source near a slowly rotating star or black hole. In § 4.1 we introduce the equation of motion for a test particle (which, by definition, does not affect the metric or the stress-energy tensor of the radiation) and use it to generalize the usual definition of the critical (radial) force outside a nonrotating star or black hole to the case of a slowly rotating object. In § 4.2 we use the generalized definition of the critical force to define and then compute critical fluxes and, where possible, critical luminosities for particles with cross sections that are independent of both the frequency and the direction of the incident radiation. In § 4.3 we discuss the behavior of critical fluxes and luminosities for particles with cross sections that depend on frequency and/or direction, illustrating such effects by two examples. Finally, in § 4.4 we discuss the maximum luminosity in general relativity of stars powered by steady, spherically symmetric radial accretion.

4.1. *Critical Radial Force*

The general relativistic equation of motion for a test particle of rest mass m may be written (see e.g., Abramowicz et al. 1990)

$$\frac{1}{m} f^\alpha = a^\alpha \,, \tag{32}$$

where f^α includes all nongravitational forces and

$$a^\alpha = \frac{d^2 x^\alpha}{d\tau^2} + \Gamma^\alpha_{\mu\nu} u^\mu u^\nu \tag{33}$$

is the acceleration measured by an accelerometer comoving with the particle. In equation (33), τ is the proper time and

$$\Gamma^\alpha_{\mu\nu} = \tfrac{1}{2} g^{\alpha\beta}(g_{\beta\mu,\nu} + g_{\beta\nu,\mu} - g_{\mu\nu,\beta}) \tag{34}$$

are the connection coefficients, where commas denote partial derivatives.

As well as describing the motion of a single particle, equation (32) may in some cases adequately describe the motion of an element of fluid. For example, near a strong X-ray source matter is likely to be fully ionized by the radiation, the electrons and ions are likely to be closely coupled by electric and magnetic fields, and the temperature may be only a few keV, due to Compton cooling. If gas-pressure–gradient forces can be neglected, and if magnetic and viscous forces are also negligible, the fluid may be treated approximately as a collection of particles with a momentum transfer cross section equal to the electron Thomson cross section and a rest mass equal to the mass per electron (see, e.g., Zel'dovich and Novikov 1971; Shapiro & Teukolsky 1983, p. 397; Miller & Lamb 1993).

We define the critical force $f_{\rm crit}$ at radius r as that radial force which keeps constant the radial velocity of a particle at r, i.e., the radial force for which

$$\frac{d^2 r}{d\tau^2} = 0 \,. \tag{35}$$

This definition makes it possible to consider the critical force not only for a particle or an element of gas in the outer part of a static star, but also for a particle or an element of gas in motion near a star or black hole.

As noted in § 3.1, the geometry of spacetime outside a slowly rotating star or black hole is unique to first order in j. The critical force for a particle in this spacetime can be computed from the radial component of the relativistic equation of motion (32) expanded to first-order in j, and the definition (35) of the critical force. The result for a particle with $u_r = u_\theta = 0$ is, in Boyer-Lindquist coordinates,

$$\frac{1}{m} f^r_{\rm crit} = \frac{M}{r^2} - \left(1 - \frac{3M}{r}\right)\frac{u_\phi^2}{r^3 \sin^2\theta} + \frac{6 j u_\phi u_t M^2}{r^4} + \mathcal{O}(j^2) \,, \tag{36}$$

where M is the gravitational mass of the object and u_ϕ and $-u_t$ are the specific angular momentum and energy of the particle. This equation is exact in u_ϕ, to $\mathcal{O}(j)$. It shows that in general the critical force depends on the angular momentum j of the gravitating mass as well as the angular momentum u_ϕ of the particle.

The term proportional to u_ϕ^2 in equation (36) shows that a particle at $r > 3M$ with $u_\phi \neq 0$ has centrifugal support against gravity, so that the critical force is less than it would be for a particle with $u_\phi = 0$. For a particle at $r < 3M$, on the other hand, the critical force is greater if $u_\phi \neq 0$ than if $u_\phi = 0$. This is related to the reversal of "inward" and "outward" at $r = 3M$ discussed by Abramowicz and collaborators (see, e.g., Abramowicz 1992 and references therein).

If u_ϕ is not small, the term in equation (36) proportional to u_ϕ^2 can strongly affect the critical force, just as in the Newtonian limit; moreover, in this case the critical force in general relatively depends on the angular momentum of the star to first order. Equation (36) shows that *to first order in j and u_ϕ (and their product), the critical force is*

$$f_{\rm crit} = \frac{mM}{r^2} \,, \tag{37}$$

and is therefore unaffected by azimuthal motion of the particle or rotation of the gravitating mass to this order. Thus, for a particle that is at rest in the LNRF ($u_\phi = 0$) or at rest in the frame that appears static as seen from infinity ($u^\phi = 0$), the effects caused by rotation of the gravitating mass and azimuthal motion of the particle are at most $\mathcal{O}(j^2)$, $\mathcal{O}(u_\phi^2)$, and $\mathcal{O}(j u_\phi)$, i.e., second-order. A particle with $u_\phi = 0$ near a rotating radiation source around a rotating mass generally experiences an azimuthal radiation force that accelerates it in the azimuthal direction, but the resulting u_ϕ is always small (Miller & Lamb 1995). A particle in the outer part of a rotating star need not have any special value of u_ϕ, but if the star is slowly rotating, u_ϕ will be small. Thus, in these situations the effects caused by rotation of the gravitating mass and azimuthal motion of the particle are at most second-order, also.

4.2. *Frequency- and Angle-independent Momentum Transfer Cross Section*

Consider now the critical (radial) radiation flux for a particle with a momentum transfer cross section that is independent of the frequency and direction of the incident radiation. The critical flux for such a particle may be calculated from expression (36) for the critical force and the relation between the radiation force and the radiation flux. Expression (36) shows that the critical flux depends on the specific angular momentum of the particle. The dependence on u_ϕ means that the critical flux varies with latitude for particles supported by gas pressure at the surface of a uniformly rotating star, just as in the Newtonian limit. The critical flux generally also depends on the angular momentum of the star, to first order, which is a purely general relativistic effect. However, if u_ϕ and j are both small, equation (37) shows that the critical flux is independent of the angular momenta of the particle and the star, to first order.

We now compute explicitly the radial component of the radiation force on a particle that has a small azimuthal velocity β relative to the LNRF. As shown in § 2.1.1, the radial component of the radiation force on the particle, in a tetrad momentarily comoving with it, is $f^{\hat{r}'}_{\rm rad} = \sigma T^{\hat{t}'\hat{r}'}$, where $T^{\hat{t}'\hat{r}'}$ is the radial component of the radiation flux measured in the comoving tetrad. The transformation (21) shows that, to first order in β and j, the radial force on the particle in the Boyer-Lindquist coordinate system is

$$f^r = e^r{}_{\mu'} f^{\mu'} = e^r{}_{\hat{r}'} f^{\hat{r}'} = (1 - 2M/r)^{1/2} f^{\hat{r}'} . \tag{38}$$

To obtain an expression for the force in terms of the radial component T^{tr} of the radiation flux in the Boyer-Lindquist coordinate system we first note that (see § 3.1)

$$T^{tr} = e^t{}_{\mu'} e^r{}_{\nu'} T^{\mu'\nu'} = T^{\hat{t}'\hat{r}'} - \frac{2jM^2}{r^3}(1 - \lambda) T^{\hat{r}'\hat{\phi}'} . \tag{39}$$

To determine $1 - \lambda$, we note that if $v^{\hat{\phi}} = \beta$, then $u^{\hat{\phi}} = \beta\gamma$ and $u^{\hat{t}} = \gamma$. Hence the angular velocity of the particle measured at radial infinity is

$$u^\phi = \frac{2jM^2\gamma}{r^3\left(1 - \frac{2M}{r}\right)^{1/2}}\left[1 + \frac{\beta r^2}{2jM^2}(1 - 2M/r)^{1/2}\right], \tag{40}$$

and so

$$\lambda = 1 + \frac{\beta r^2}{2jM^2}\left(1 - \frac{2M}{r}\right)^{1/2} . \tag{41}$$

Substituting this expression for λ into equation (39), we find, to first order in β,

$$T^{tr} = T^{\hat{t}'\hat{r}'} + \frac{\beta}{r}\left(1 - \frac{2M}{r}\right)^{1/2} T^{\hat{r}'\hat{\phi}'} . \tag{42}$$

Now $T^{\hat{r}'\hat{\phi}'}$ is related to the stress-energy $T^{\mu\nu}$ in the LNRF by a Lorentz boost by β in the ϕ-direction; using the Lorentz transformation (19) gives, to first order in β,

$$T^{\hat{r}'\hat{\phi}'} = T^{\hat{r}\hat{\phi}} - \beta T^{\hat{t}\hat{\phi}} . \tag{43}$$

Finally, substituting this result into equation (42) gives

$$T^{tr} = T^{\hat{t}'\hat{r}'} + \frac{\beta}{r}\left(1 - \frac{2M}{r}\right)^{1/2}(T^{\hat{r}\hat{\phi}} - \beta T^{\hat{t}\hat{\phi}})$$

$$= T^{\hat{t}'\hat{r}'} + \frac{\beta}{r}\left(1 - \frac{2M}{r}\right)^{1/2} T^{\hat{r}\hat{\phi}} + \mathcal{O}(\beta^2) . \tag{44}$$

As discussed in § 3.3, $T^{\hat{r}\hat{\phi}}$ is first-order in j and v, so the radial flux $F(n^r) \equiv T^{tr}$ in the Boyer-Lindquist coordinate system is equal to the flux $T^{\hat{t}'\hat{r}'}$ in the comoving tetrad, plus terms $\mathcal{O}(j^2)$, $\theta(\beta^2)$, $\mathcal{O}(vj)$, $\mathcal{O}(\beta v)$, and $\mathcal{O}(\beta j)$. Thus, to first order in j, v, and β, the radial component of the radiation force may be written

$$f^r_{\rm rad} = (1 - 2M/r)^{1/2} \sigma F(r; n^r) , \tag{45}$$

in terms of quantities expressed in the Boyer-Lindquist coordinate system. Equation (45) agrees, to first order, with the expression one obtains by evaluating equation (15).

The critical radial flux for a particle at r, can be determined by setting the radiation force (45) equal to the critical force (37); the result is

$$F_{{\rm crit},r}(r'; n^{r'}) = \frac{(1 - 2M/r)^{1/2}}{(1 - 2M/r')} \frac{M}{r^2}\left(\frac{m}{\sigma}\right) . \tag{46}$$

This critical flux is the same as in the absence of rotation. Therefore, we conclude that *if the momentum transfer cross section is independent of both angle and frequency, the critical flux for a slowly orbiting particle near a slowly rotating mass and radiation source is the same, to first order in the rotation rates, as if the particle, mass, and source were not rotating.*

In § 3.3 we showed that if the radiation source is axisymmetric about its rotation axis and the emission is back–front symmetric, as expected for emission from a rotating star or an accretion disk or boundary layer around a star or black hole, the radial component of the radiation flux at r is independent of the rotation rates of the star and the radiation source, to first order in j and v. Thus, we can conclude not only that the critical radial flux for a slowly orbiting particle at r, as measured at r, is the same, to first order, as it would be if the particle, mass, and source were not rotating, but also that *the radial flux at the source that produces a critical flux at r is the same, to first order, as it would be if the particle, mass, and source were not rotating.*

An important corollary is that *the critical flux for a particle moving in a circular Keplerian orbit in the rotation equator of the gravitating mass is exactly the same as it would be for a static particle at the same radius.* The reason is that as the flux approaches the critical flux, the specific angular momentum of a particle in circular Keplerian orbit approaches zero, due to the radial support provided by the radiation force, and the terms in the radial equation of motion that involve u_ϕ therefore vanish.

Given these results, what is the best choice for a benchmark flux for rotating astrophysical systems? For a system in which the gravitational mass is spherically symmetric and static, the critical flux at radius r that is most useful as an astrophysical benchmark is the critical flux for a particle that is at rest at r in the static frame (i.e., the frame that is at rest with respect to infinity). In contrast, if the gravitational mass is rotating there is no local property of spacetime that singles out particles that are at rest as seen from infinity (i.e., particles with $\Omega = d\phi/dt = 0$); indeed, in general there is a region of spacetime, the ergosphere, where a particle cannot be at rest with respect to infinity (Bardeen et al. 1972).

Our results show that the critical flux for matter at rest in the LNRF will often be satisfactory as an astrophysical benchmark flux. The critical fluxes for matter at rest in the static frame (where this is possible), in Keplerian orbit around the gravitating mass, or in the outer part of a slowly rotating star are all equal to the critical flux for matter at rest in the LNRF, at least to first order. Moreover, if the critical flux for matter at rest in the LNRF is adopted as the astrophysical benchmark flux, many potential complications introduced by rotation of the radiation source and the gravitating mass are avoided: the radial components of the radiation flux in the LNRF and in Boyer-Lindquist coordinates are equal, to first order in j (§ 3.1), while for matter that is at rest in the LNRF, the change in the critical flux caused by the rotation of the mass is zero, to first order in j (§ 4.1). *We therefore suggest that for a system with a rotating gravitational mass, the critical radiation flux that is likely to be most useful as an astrophysical benchmark is the critical flux for matter at rest in the LNRF.*

In many astrophysical situations it is the critical luminosity (which is a global quantity) rather than the critical flux (a local quantity) that is of interest as a benchmark. For the critical luminosity to be well-defined, the radial component of the radiation flux must be the same in all directions; otherwise, for a given luminosity the radiation force could be subcritical in some directions and supercritical in others. Fortunately, many astrophysical objects produce radiation fluxes that are approximately spherically symmetric.

In § 3 we showed that if the radiation flux produced by a system composed of a nonrotating mass and radiation source is spherically symmetric, the flux remains spherically symmetric to first order in the rotation rates when the mass and source are rotating. As shown above, the critical flux for a particle with specific angular momentum u_ϕ remains unchanged to first order in j, v, u_ϕ. Combining these two results, we find that the critical luminosity for a slowly orbiting particle at r, as measured at r', is

$$L_{\mathrm{crit},r}(r') = 4\pi r^2 F_{\mathrm{crit},r}(r'; n^{r'}) = \frac{(1 - 2M/r)^{1/2}}{(1 - 2M/r')} L_{\mathrm{E}} , \quad (47)$$

so that the critical luminosity for a particle at r, as measured at infinity, is

$$L_{\mathrm{crit},r}(\infty) = (1 - 2M/r)^{1/2} L_{\mathrm{E}} . \quad (48)$$

Here

$$L_{\mathrm{E}} \equiv 4\pi M(m/\sigma) \quad (49)$$

is the critical luminosity in the Newtonian limit. The critical luminosities (47) and (48) are the same as in the absence of rotation (see, e.g., Shapiro & Teukolsky 1983, p. 396). Thus, *if the momentum transfer cross section is independent of both angle and frequency, the critical luminosity for a slowly orbiting particle near a slowly rotating mass and radiation source is the same, to first order in the rotation rates, as if the particle, mass, and source were static.* In this case the critical luminosity is also independent of the spectrum and angular size of the radiation source.

In the Newtonian limit the critical luminosity does not depend on the location of the particle or the radius at which the luminosity is measured, provided that the cross section is independent of frequency and direction. The simplicity and robustness of this result make L_{E} a good benchmark for luminous sources. The critical luminosity in general relativity differs from the critical luminosity in the Newtonian limit in that, for a particle at radius r, the critical luminosity measured at r is a factor $(1 - 2M/r)^{-1/2}$ larger than the Newtonian value, while the critical luminosity measured at infinity is a factor $(1 - 2M/r)^{1/2}$ smaller. Thus, in general relativity the critical luminosity depends both on the radial position of the particle and on the radius at which the luminosity is measured, even if the cross section is frequency- and angle-independent. However, as we have shown here, the critical luminosity in general relativity is robust in the sense that it is unaffected, to first order, by rotation of the particle, mass, and source. Also, the luminosity at the source that produces a critical luminosity at r is the same, to first order, as it would be if the particle, mass, and source were not rotating.

As discussed in § 2.3, the results presented here for a test particle are also valid under fairly general conditions for an element of gas, such as an electron-ion plasma, if m/σ is replaced by $1/\kappa$, where κ is the opacity computed from the relevant cross sections. As explained in § 2.3, even if the relevant cross sections are angle-dependent, the opacity is generally direction-independent unless particles are oriented or the angle dependence of the cross sections is related to some special direction within the system, such as that defined by a magnetic field. If in addition the opacity is frequency-independent, the results derived in this section apply. Radial dependence of the opacity can be accommodated by replacing m/σ by $1/\kappa(r)$. Thus these results are more generally relevant than they might at first seem to be.

4.3. *Frequency- and/or Angle-dependent Momentum Transfer Cross Section*

Next consider the critical (radial) radiation flux for a particle with a momentum transfer cross section that depends on the frequency or direction of the incident radiation. As shown in § 2.1.1, the radial component of the force in a tetrad momentarily comoving with the particle is $f^{\hat{r}}_{\mathrm{rad}} = \langle\sigma(n^{\hat{r}})\rangle F(n^{\hat{r}})$, where the appropriately angle-averaged cross section is given by equation (7). The argument leading to equation (46) is still valid if σ is replaced by $\langle\sigma\rangle$. However, unlike in our previous analysis, here $\langle\sigma\rangle$ itself is generally a function of r.

As a simple example, suppose the cross section varies with direction; then the variation of the apparent angular size of the source with the particle's distance from it will introduce a dependence of $\langle\sigma\rangle$ on r. As a result of the r-dependence of $\langle\sigma\rangle$, the expression for the critical flux in this case has an r-dependence that is different from the r-dependence of the standard Newtonian and general relativistic expressions for the critical flux, which assume that the cross section is frequency- and angle-independent.

In general, computation of $\langle\sigma\rangle$ requires tracing of rays from the radiation source to the position of the particle. If, however, the radiating matter is all at the same radius, the specific intensity at the radiation source is independent of direction (i.e., the intensity distribution is isotropic), and the cross section is independent of frequency, $\langle\sigma\rangle$ is easily computed without ray-tracing, by using the relation $I \propto \nu^4$ discussed in § 3.2. The reason is that in this case the actual value of the specific intensity at r does not enter the computation of $\langle\sigma\rangle$, which reduces to a simple average of the cross section over the solid angle subtended by the source at the position of the particle.

To illustrate the effect on the critical flux of cross section angle-dependence, we consider particles with the two frequency-independent but angle-dependent cross sections discussed in § 2.1.2, outside a nonrotating, uniformly emitting, spherical source. To simplify the calculation, we assume that the specific intensity at the source is independent of direction. Then as just discussed, the angle average reduces to

$$\langle\sigma\rangle = \frac{\int_0^{2\pi}\int_0^{\alpha_0} \sigma(\tilde{a}, \tilde{b}) \cos\tilde{a} \sin\tilde{a}\, d\tilde{a}\, d\tilde{b}}{\int_0^{2\pi}\int_0^{\alpha_0} \cos\tilde{a} \sin\tilde{a}\, d\tilde{a}\, d\tilde{b}} , \quad (50)$$

where α_0 is given by equation (27). For a flat particle with faces of area σ_0 oriented perpendicular to the radial direction,

$$\langle \sigma \rangle = \frac{2}{3}\left(\frac{1 - |\cos^3 \alpha_0|}{\sin^2 \alpha_0}\right)\sigma_0 , \quad (51)$$

and hence the critical flux for a particle at r, as measured at r', is

$$F_{\mathrm{crit},r}(r') = \frac{(1 - 2M/r)^{1/2}}{(1 - 2M/r')} \frac{M}{r^2} \frac{3}{2}\left(\frac{\sin^2 \alpha_0}{1 - |\cos^3 \alpha_0|}\right)\frac{m}{\sigma_0} \quad \text{(case 1)} , \quad (52)$$

where the subscript on the flux indicates the position of the particle and its argument indicates the position at which the flux is measured. If instead the faces of the particle are oriented perpendicular to the ϕ-direction,

$$\langle \sigma \rangle = \left(\frac{4 \sin \alpha_0}{3\pi}\right)\sigma_0 \quad (53)$$

and hence the critical flux for a particle at r, as measured at r', is

$$F_{\mathrm{crit},r}(r') = \frac{(1 - 2M/r)^{1/2}}{(1 - 2M/r')} \frac{M}{r^2}\left(\frac{3\pi}{4 \sin \alpha_0}\right)\frac{m}{\sigma_0} \quad \text{(case 2)} . \quad (54)$$

Since the radiation field is spherically symmetric, the critical luminosities are well-defined and are

$$L_{\mathrm{crit},r}(r') = \frac{(1 - 2M/r)^{1/2}}{(1 - 2M/r')} \frac{3}{2}\left(\frac{\sin^2 \alpha_0}{1 - |\cos^3 \alpha_0|}\right)L_{\mathrm{E}} \quad \text{(case 1)}$$

and (55)

$$L_{\mathrm{crit},r}(r') = \frac{(1 - 2M/r)^{1/2}}{(1 - 2M/r')}\left(\frac{3\pi}{4 \sin \alpha_0}\right)L_{\mathrm{E}} \quad \text{(case 2)} , \quad (56)$$

where L_{E} is given by the usual expression for the critical luminosity in the Newtonian limit (eq. [49]), with σ replaced by σ_0. In case 2, the effects of the redshift, time-dilation, and stronger radial gravitational force in general relativity exactly offset the change in the apparent angular size of the source, so that $L_{\mathrm{crit},r}(\infty) \propto r$ in general relativity as well as in the Newtonian limit.

The critical luminosities for case 1 and case 2 are shown as functions of r in Figure 3. Also shown is the critical luminosity for an angle-independent cross section equal to σ_0. These plots show that the effect of cross section angle dependence can be quite large. In case 1, the critical luminosity is 50% greater than the standard Newtonian or general relativistic critical luminosities at the source (where $\alpha = \pi/2$). In case 2, the critical luminosity is 2.4 times larger than the standard luminosities at the source, and becomes arbitrarily large as $r \to \infty$ and $\alpha \to 0$.

These examples show that if the cross section is angle-dependent, the apparent angular size of the radiation source generally enters the expression for the critical flux, changing its dependence on the radial position r of the particle. As a result, *the critical luminosity generally depends on r even in the Newtonian limit.* The r-dependence of the critical luminosity in general relativity is different from that in the Newtonian limit because the apparent angular size of the source is affected by gravitational lensing while the radiation flux is affected by redshift and time-dilation. Moreover, in general relativity the apparent angular size of the source varies with radial position

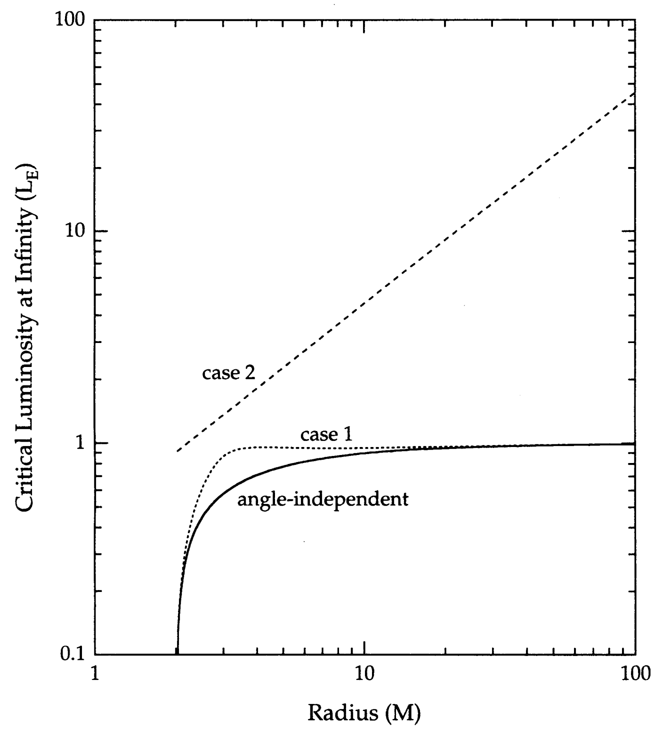

FIG. 3.—Critical luminosities for particles as a function of their radial position r, as measured at infinity in units of L_{E}, the critical luminosity in the Newtonian limit, for spherically symmetric emission from a spherical surface of radius $2.01M$. Shown are the critical luminosities for a particle with an angle-independent cross section (*solid line*) and oriented particles with the two angle-dependent cross sections referred to in the text as case 1 (*dotted line*) and case 2 (*dashed line*).

differently for source radii larger and smaller than $3M$; therefore, if the cross section is angle-dependent, the variation of the critical luminosity with r depends on whether the radius of the source is greater or less than $3M$ (see eq. [27]).

Although the derivation given here assumes a nonrotating mass and radiation source, for the two particle orientations we have considered the critical fluxes and luminosities are not affected, to first order, if the mass and radiation source are rotating. However, in general the critical fluxes and luminosities *are* affected to first order. More generally, the cross section may depend on frequency and direction in a correlated way, in which case the critical flux will depend in a correlated way on the spectrum of the radiation source as well as its angular size.

Equation (55) is (in our notation) the expression presented (incorrectly) by Nemiroff et al. (1993) as the critical luminosity for a Thomson-scattering particle.[8] Our analysis shows that the difference between their expression and the usual expression (47) is due entirely to their use of an incorrect expression for the force on the particle. As discussed in § 2.1.3, their expression for the force is equivalent to assuming that the cross section for Thomson scattering is $\sigma(\mathbf{n}) = \sigma_{\mathrm{T}} |\cos \tilde{\alpha}|$ rather than σ_{T}, i.e., that the electron behaves like a flat, infinitesimally thick, perfectly absorbing surface of area σ_{T}, oriented so that its normal points in the local radial direction. This is

[8] Their expression for the critical luminosity is given by their eqs. (7) and (8). Nemiroff et al. denote the Schwarzschild coordinate radius of the particle by r_{obs} and use ψ where we use α to denote the half-angle of the limb of the source.

responsible for the erroneous conclusion of Nemiroff et al. that the critical luminosity for Thomson scattering depends on the angular size of the source.

As in the case of frequency- and angle-independent cross sections, the results presented in this section are valid under fairly general conditions for an element of gas, if m/σ is replaced by $1/\kappa(n^{\hat{r}})$, where $\kappa(n^{\hat{r}})$ is the opacity computed from the appropriately angle-, frequency-, and particle-averaged cross sections (see § 2.3).

4.4. *Critical Luminosity for Radial Accretion*

So far we have considered the critical fluxes and luminosities that keep the radial velocity of a particle constant. They therefore cause a particle that is at radius r and initially has no radial velocity ($u_r = 0$) to remain at r. In computing these fluxes and luminosities, we allowed the gravitating mass and the radiation source to be surrounded by a vacuum or by absorbing or scattering matter, but we always assumed that the luminosity of the system was unaffected by such matter. We now consider the maximum luminosity of a nonrotating star that is powered, at least in part, by spherically symmetric radial accretion. In this case the accreting matter outside the radiation source generally does affect the luminosity of the system. In such a system, the matter outside the star is flowing radially inward, so that $u_r \neq 0$.

We assume that the radiation field is spherically symmetric. Because the critical luminosity (47) decreases with radius in general relativity, one might expect that inflow could begin at some large radius, where the luminosity is subcritical, only to stall at a smaller radius, where the luminosity has become supercritical. As we show, generalizing an argument due to Park & Miller (1991), this is possible, but only if the fraction of the total luminosity produced by the accretion flow is less than the efficiency $\epsilon = 1 - (1 - 2M/R)^{1/2}$ of the system in producing radiation from a unit mass of accreted matter (for a neutron star R is the stellar radius, whereas for radial flow onto a black hole R is the smallest radius at which luminosity is produced). The reason is that the infalling matter performs work on the escaping radiation. As a result, the luminosity increases with increasing radius and, unless the accretion luminosity is a sufficiently small fraction of the total, $L_r(\infty)/L_{\mathrm{crit},r}(\infty)$ is greatest at radial infinity and hence the critical luminosity for steady flow in general relativity, as measured at infinity, is L_{E}, the same as in the Newtonian approximation.

To see this, we begin by noting that, as shown by the detailed study of Park & Miller (1991), $|u_r|$ is small everywhere throughout the flow when the luminosity approaches its critical value. If $|u_r|$ were zero, the radiation flux $F_{\mathrm{co}}(r)$ measured in the local frame comoving with the matter would be equal to the stationary-frame flux $F(r)$, and the luminosity would be critical when, somewhere in the flow, $F(r)$ became equal to the local critical flux $F_{\mathrm{crit},r}(r)$. Since $|u_r|$ is not zero, the luminosity is critical when $F_{\mathrm{co}}(r)$ exceeds the local critical flux $F_{\mathrm{crit},r}(r)$ by an amount sufficient to halt the inflow. The precise condition for criticality cannot be determined without a detailed, global flow solution. However, Park & Miller (1991) have shown that $|u_r|$ is very small near the critical radius r_{crit} where $F_{\mathrm{co}}(r)$ first equals $F_{\mathrm{crit},r}(r)$. Therefore, it is a good approximation to assume that the luminosity is critical when $F_{\mathrm{co}}(r)$ equals $F_{\mathrm{crit},r}(r)$ somewhere in the flow. Since $F(r)$ bounds $F_{\mathrm{co}}(r)$ from below, this implies that steady inflow is not possible if $F(r) > F_{\mathrm{crit},r}(r)$ somewhere in the flow. Hence, we can determine the maximum luminosity of the system by computing the accretion luminosity $L_{\mathrm{accr}} = \epsilon \dot{M}$, or equivalently, the radial mass flux $\dot{M}$, for which $F(r)$ first equals $F_{\mathrm{crit},r}(r)$ somewhere in the flow.

For simplicity, we continue the argument in terms of luminosities rather than fluxes and measure all luminosities in a static frame at radial infinity. The maximum luminosity of the system will be equal to the critical luminosity for matter at infinity, $L_{\mathrm{crit},\infty}(\infty)$, if and only if $L_{\mathrm{crit},r}(\infty) \geq L_r(\infty)$ for all finite r when $L_\infty(\infty) = L_{\mathrm{crit},\infty}(\infty)$. The luminosity generated between radial infinity and r by interaction of the accretion flow with the radiation field is $[1 - (1 - 2M/r)^{1/2}]\dot{M}$ (the kinetic energy terms that would normally appear in this expression have been neglected, because the radial inflow is slow everywhere). Therefore, when $L_\infty(\infty) = L_{\mathrm{crit},\infty}(\infty)$, $L_r(\infty) = L_{\mathrm{crit},\infty}(\infty) - [1 - (1 - 2M/r)^{1/2}]\dot{M}$, so the condition $L_{\mathrm{crit},r}(\infty) \geq L_r(\infty)$ can be written

$$L_{\mathrm{crit},r}(\infty) \geq L_{\mathrm{crit},\infty}(\infty) - [1 - (1 - 2M/r)^{1/2}]\dot{M} \quad (\text{all } r < \infty) . \tag{57}$$

Given any $L_{\mathrm{crit},r}(\infty)$ (such as that for a direction- or radius-dependent opacity), inequality (57) can be used to determine whether the maximum luminosity is equal to the critical luminosity at radial infinity.

Suppose now that the opacity is direction- and radius-independent. Then $L_{\mathrm{crit},\infty}(\infty) = L_{\mathrm{E}}$, $L_{\mathrm{crit},r}(\infty) = (1 - 2M/r)^{1/2} L_{\mathrm{E}}$, and condition (57) becomes

$$(1 - 2M/r)^{1/2}(L_{\mathrm{E}} - \dot{M}) \geq L_{\mathrm{E}} - \dot{M} \quad (\text{all } r < \infty) . \tag{58}$$

This condition is satisfied if $\dot{M} \geq L_{\mathrm{E}}$, i.e., if the accretion flow produces at least the fraction ϵ of the total luminosity. For typical neutron star masses and equations of state, $\epsilon \sim 0.2$, so condition (58) will generally be satisfied if more than $\sim 20\%$ of the luminosity is produced by the radial accretion flow. In accreting neutron star systems, a radial flow component that generates this fraction of the total luminosity is expected to be created by azimuthal radiation drag, when the luminosity is near-critical (Lamb 1989, 1991b, 1994; Fortner, Lamb, & Miller 1989, 1994).

To see what happens when $\dot{M} < L_{\mathrm{E}}$, we need to investigate the variation of $L_{\mathrm{crit},r}(\infty) - L_r(\infty) = (1 - 2M/r)^{1/2} L_{\mathrm{E}} - L_r(\infty)$ with radius, which is shown by its derivative

$$\frac{d}{dr}\,[L_{\mathrm{crit},r}(\infty) - L_r(\infty)] = (L_{\mathrm{E}} - \dot{M})\left(1 - \frac{2M}{r}\right)^{-1/2} \frac{M}{r^2} . \tag{59}$$

Until the critical luminosity is reached somewhere, $L_{\mathrm{crit},r}(\infty) - L_r(\infty)$ is positive everywhere. Also, the factor $(1 - 2M/r)^{-1/2}$ on the right is always positive outside $2M$. Therefore, if $\dot{M} > L_{\mathrm{E}}$, the derivative of $L_{\mathrm{crit},r}(\infty) - L_r(\infty)$ is negative, showing that the stationary-frame luminosity gets closer to the critical luminosity as r increases, and is closest at infinity, confirming our earlier result. If on the other hand $\dot{M} < L_{\mathrm{E}}$, then the derivative of $L_{\mathrm{crit},r}(\infty) - L_r(\infty)$ is negative, showing that the stationary-frame luminosity gets closer to the critical luminosity as r decreases, and is closest at the stellar surface, so it is possible for matter to begin to accrete from infinity and then to stall closer to the star, causing the flow to become time-dependent. If $\dot{M} < L_{\mathrm{E}}$, the maximum luminosity for steady spherical inflow, as seen at infinity, is $L_{\max}(\infty) = (1 - 2M/R)^{1/2} L_{\mathrm{E}} + \epsilon \dot{M}$.

5. RESULTS AND CONCLUSIONS

The critical flux (or luminosity) at which the outward force of radiation balances the inward force of gravity plays an important role in many astrophysical systems. In this work we have studied the radiation force on particles with arbitrary cross sections and investigated the radiation field produced by radiating matter, such as a disk, ring, or stellar surface, that rotates slowly around a slowly rotating, gravitating mass. We then used the results to obtain expressions for the critical radiation flux and, where appropriate, the critical luminosity, both when the opacity is frequency- and direction-independent and when it is frequency- or direction-dependent. As discussed in § 1, all neutron stars with known periods rotate slowly in the sense referred to here. Black holes in binary systems are also expected to be slowly rotating in this sense, and some black holes in active galactic nuclei may be slowly rotating.

Here we summarize our results; details may be found in the cited subsections.

5.1. *Radiation Force*

In § 2 we analyzed the radiation force on a scattering or absorbing particle and showed that:

1. The radiation force on a particle or an element of gas is not proportional to the diagonal ("pressure") components of the radiation stress-energy tensor nor is it in general proportional to *any* component of the radiation stress-energy tensor, including the radiation flux. In particular, if the cross section depends on the frequency but not the direction of the incident radiation, as in Compton scattering, the radiation force generally is neither proportional to nor in the direction of the radiation flux (§ 2.1).
2. If the momentum transfer cross section is independent of both the frequency and the direction of the incident radiation, the radiation force is in the direction of the local radiation flux measured in an orthonormal tetrad comoving with the particle, with a magnitude that is proportional to the radiation flux (§ 2.1).
3. The radiation force on a small volume of gas is equal to the sum of the forces on the constituent particles under fairly general conditions (§ 2.3). The momentum transfer cross sections of the constituent particles may be replaced by the cross sections of particles at rest in the center-of-momentum frame of the gas if the thermal and drift velocities of the particles are not too large (§ 2.3). Even if the cross sections of the constituent particles are angle-dependent, the opacity of an element of gas is independent of direction unless the particles are oriented or the angle dependence of the cross sections is related to some direction in the system, such as that defined by a magnetic field. Only if the opacity is independent of both frequency and direction, the relevant particle speeds are small compared to c, and the photon mean free path is small compared to other relevant distances is the force on an element of gas proportional to the *gradient* of the radiation pressure (§ 2.3).

5.2. *Radial Component of the Radiation Flux*

In § 3.2 we analyzed the radial component of the radiation flux measured in a static tetrad near a nonrotating black hole or relativistic star that produces a time-independent and spherically symmetric radiation field and showed that it is independent of the radius and angular size of the radiation source. The flux is also independent of the optical depth of any matter that may surround the radiation source, provided the matter is distributed spherically symmetrically and does not alter the luminosity.

In § 3.3 we investigated the radiation field produced by radiating matter, such as a disk ring, or stellar surface, that rotates slowly around a slowly rotating gravitating mass. We showed that if the rotation axes of the radiating matter and the gravitating mass are co-aligned (the radiating matter and the mass need not rotate at the same rate, or in the same sense), then:

1. Doppler and frame-dragging effects causes changes in the radiation field that are first-order in the appropriately averaged azimuthal velocity v of the matter in the radiation source and the dimensionless angular momentum j of the rotating mass. If the angular velocity of the radiation source is similar to or greater than the angular velocity of the gravitating mass, the Doppler effects are generally much larger than the effects of frame-dragging.
2. If the radiation source has symmetries that are expected for many astrophysical sources, then the angular shape of the radiation source, the radial component of the radiation flux, and all of the diagonal elements of the stress-energy tensor measured in a slowly rotating tetrad are the same, to first order in v, j, and the velocity of the tetrad, as they would be if the source, mass, and tetrad were not rotating. Since $j \lesssim 0.2$ for neutron stars with measured spin periods, the effects of rotation are at most $\mathcal{O}(j^2) \sim 4\%$ for these stars.

5.3. *Critical Fluxes and Luminosities*

If the radiation field produced near a star or black hole is asymmetric, as it would be if radiation is emitted only from a thin band around the equator (perhaps because the object is accreting matter from a geometrically thin disk), there is no obvious definition of the critical luminosity nor, supposing a definition were adopted, would it be likely to prove useful as a general benchmark. Fortunately, many astrophysical systems produce radiation fields that are approximately spherically symmetric. If the flux produced by a system would be spherically symmetric in the absence of rotation, it remains spherically symmetric to first order in j and v if the mass and source are rotating. Thus one can define a critical luminosity for such a system that is accurate to first order.

In general, the critical flux and luminosity depend not only on the specific angular momentum of the matter, but also on the angular momentum of the gravitating mass. However, if both u_ϕ and j are small, the dependence is only second-order [$\mathcal{O}(j^2)$, $\mathcal{O}(ju_\phi)$, and $\mathcal{O}(u_\phi^2)$] (§ 4.2). Our results for this case may be summarized as follows.

Frequency- and direction-independent opacity.—If the opacity is independent of both the frequency and the direction of the incident radiation, then:

1. The critical radial flux for slowly rotating matter is given by equation (46) and is the same, to first order in the specific angular momentum u_ϕ of the matter, the angular momentum j of the mass, and the appropriately averaged azimuthal velocity v of the radiation source, as it would be if the matter, mass, and source were not rotating. The critical flux for matter in circular Keplerian orbit in the rotation equator of the gravitating mass is *exactly* the same as it would be for static matter at the same radius, since $u_\phi = 0$ for such an orbit if the flux is critical (§ 4.2.1).
2. The critical luminosity for slowly rotating matter in a system that would produce a spherically symmetric radiation field in the absence of rotation is given by equation (48) and is the same, to first order in u_ϕ, j, and v, as it would be if the matter, mass, and source were not rotating. Unlike in the Newtonian limit, in general relativity the critical luminosity varies

with the radial location of the matter and the radius at which the luminosity is measured (§ 4.2.1).

3. As in the Newtonian approximation, the critical luminosity in general relativity is independent of the angular size as well as the spectrum of the radiation source. Thus, if the radiation field produced by the system is spherically symmetric, a spherically symmetric distribution of absorbing or scattering material between the source and the matter in question that does not change the luminosity will not change the critical luminosity at the matter or the luminosity at the radiation source that produces the critical luminosity at the matter. The conclusion of Nemiroff et al. (1993) that the critical luminosity for Thomson scattering depends on the angular size of the radiation source (see also Walker 1992) is due entirely to their use of an incorrect expression for the radiation force on a Thomson-scattering particle (§ 4.2.1).

4. Our results show that the critical flux for matter at rest in the LNRF will often be satisfactory as an astrophysical benchmark flux, since the critical fluxes for matter at rest in the static frame (where this is possible), in Keplerian orbit around the gravitating mass, or in the outer part of a slowly rotating star are all equal to the critical flux for matter at rest in the LNRF, at least to first order (§ 4.2). Moreover, if the critical flux for matter at rest in the LNRF is adopted as the astrophysical benchmark flux, many of the complications potentially introduced by rotation of the radiation source and the gravitating mass are avoided (§§ 3.1 and 4.1). *We therefore suggest that for a system with a rotating gravitational mass, the critical radiation flux for matter at rest in the LNRF should be used as the astrophysical benchmark.*

Frequency- and/or direction-dependent opacity.—If the opacity depends on either the frequency or the direction of the incident radiation, as in Compton scattering, then:

1. The critical flux generally *differs* by amounts that are first-order in u_ϕ, j, and v, from what it would be if the matter, mass, and source were not rotating. However, for neutron stars with measured periods, $j < 0.2$, so even in this case the effects of rotation are at most only $\sim 20\%$ (§ 4.2.2).

2. Even if the flux produced by a slowly rotating system is spherically symmetric to first order in j and v, the luminosity that produces a critical flux in one direction generally differs from the luminosity that produces a critical flux in another by amounts that are first-order in u_ϕ, j, and v. Therefore, one generally cannot define a critical luminosity that is accurate to first order (§ 4.2.2).

3. If the radiation field is spherically symmetric and the matter, mass, and source are all static, one can define a critical luminosity. However, the critical luminosity generally depends on the angular size and spectrum of the radiation source, both in the Newtonian approximation as well as in general relativity. If so, the critical luminosity will depend on the radial location of the matter even in the Newtonian limit (§ 4.2.2). Moreover, a spherically symmetric distribution of matter between the source and the matter in question will generally change the critical flux by, for example, altering the angular brightness distribution or the spectrum of the radiation source seen by the matter in question, even if it does not change the luminosity. These effects may be important in some systems, such as active galactic nuclei, that have hard spectra and copious numbers of electrons.

Although direction dependence of the opacity can strongly affect the critical luminosity, this complication is rarely of practical importance in situations where the critical luminosity is relevant. Even if the relevant particle cross sections are angle-dependent, the opacity of a gas of particles is generally direction-independent, unless the particles are oriented for some reason or the angle dependence of the cross section is related to some special direction within the system (§ 2.3). An example of the latter is Compton scattering in a strong magnetic field, which leads to an opacity that depends on the angle between the direction of the incident radiation and the direction of the magnetic field. However, if the magnetic field is strong enough to affect cross sections, magnetic forces are likely to be dynamically important, in which case a critical luminosity defined in terms of gravitational and radiation forces alone probably will not be very useful.

5.4. *Spherically Symmetric Accretion*

In § 4.4 we considered the maximum luminosity of a nonrotating star powered, at least in part, by spherically symmetric raidal accretion. Generalizing slightly an argument of Park & Miller (1991), we derived a condition on the critical luminosity as a function of radius that, if satisfied, guarantees that the critical luminosity of the system is equal to the critical luminosity for matter at infinity.

We then showed that if the opacity is direction- and radius-independent and $\dot{M}c^2 \geq L_E$, the maximum luminosity for steady inflow is exactly the classical Eddington luminosity L_E. This condition on $\dot{M}$ is equivalent to the statement that the radial accretion flow accounts for at least a fraction ϵ of the total luminosity, where ϵ is the efficiency of gravitational mass in producing radiation from accreting mass. For neutron stars, $\epsilon \sim 0.2$, so the critical luminosity is L_E if the radial accretion flow accounts for at least 20% of the luminosity. This is thought to be the case, for example, in the luminous neutron star sources called "Z sources," when they are on the normal branch (Lamb 1989, 1991b, 1994; Fortner et al. 1989, 1994).

If instead $\dot{M}c^2 < L_E$ (or, equivalently, if the radial accretion flow accounts for less than a fraction ϵ of the total luminosity), the luminosity first becomes critical at the stellar surface, so it is possible for matter to begin to accrete from infinity and then to stall closer to the star causing the flow to become time-dependent. In this case the maximum luminosity for steady spherical inflow is $(1 - 2M/R)^{1/2}L_E + \epsilon\dot{M}$, where R is the stellar radius.

5.5. *Conclusion*

Despite many potential complications, the critical luminosities of systems composed of a slowly rotating black hole or relativistic star and a rotating radiation source are usually accurately given by the expressions for the critical luminosities of systems composed of static masses and sources. The critical luminosity for relativistic stars and black holes powered by radial accretion is exactly the classical Eddington critical luminosity. Only when, as in Compton scattering, the opacity is strongly frequency-dependent are the effects of rotation likely to be appreciable. Even in this case the effects are $\lesssim 20\%$ for slowly rotating systems.

It is a pleasure to thank Igor Novikov for numerous helpful comments and for reading critically an early draft of the paper. We also thank Guy Miller for helpful discussions of the maximum luminosity in radial accretion and Susan Lamb for several helpful suggestions. This work was completed while both authors were visiting NORDITA. We are grateful to the staffs of NORDITA and the Niels Bohr Institute and especially the director of NORDITA, Christopher Pethick, for their warm hospitality and help during our visit. F. K. L. also thanks the staff of the Observatory of the University of Copenhagen and especially its director, Henning Jørgensen, for their assistance and hospitality. This work was supported in part by NSF grants PHY 91-00283 and AST 93-15133 and NASA grant NAGW 1583 at the University of Illinois; NASA grant NAGW 830 at the University of Chicago; and NORDITA. F. K. L. gratefully acknowledges support from the Danish Research Academy. M. C. M. gratefully acknowledges support from NASA through the *Compton* Fellowship Program (grant NAS 5-28543).

REFERENCES

Abramowicz, M. A. 1992, MNRAS, 256, 710
Abramowicz, M. A., Ellis, G. F. R., & Lanza, A. 1990, ApJ, 361, 470
Bardeen, J. M. 1970a, Nature, 226, 64
———. 1970b, ApJ, 162, 71
Bardeen, J. M., Press, W. H., & Teukolsky, S. A. 1972, ApJ, 178, 347
Begelman, M. C., Blandford, R. D., & Rees, M. J. 1984, Rev. Mod. Phys., 56, 255
Berestetskiĭ, V. B., Lifshitz, E. M., & Pitaevskiĭ, L. P. 1971, Relativistic Quantum Theory, P.1 (Oxford: Pergamon Press)
Bjornsson, G., & Svensson, R. 1992, ApJ, 394, 500
Blandford, R. D. 1990, in Active Galactic Nuclei, ed. T. J.-L. Courvoisier & M. Mayor (Berlin: Springer-Verlag), 161
Blandford, R. D., & Rees, M. J. 1992, in Testing the AGN Paradigm, ed. S. S. Holt, S. G. Neff, & C. M. Urry (AIP Conf. Proc. 254), 3
Bowers, R. L., & Deeming, T. 1984, Astrophysics, Vol. 1 (Boston: Jones & Bartlett)
Boyer, R. H., & Lindquist, R. W. 1967, J. Math. Phys., 8, 265
Bussard, R. W., Alexander, S. B., & Mészáros, P. 1986, Phys. Rev. D, 34, 440
Cook, G. B., Shapiro, S. L., & Teukolsky, S. A. 1992, ApJ, 398, 203
———. 1994, ApJ, 424, 823
Czerny, M., Czerny, B., & Grindlay, J. E. 1986, ApJ, 311, 241
Daugherty, J. K., & Harding, A. K. 1986, ApJ, 309, 362
Done, C., & Fabian, A. C. 1989, MNRAS, 240, 81
Eddington, A. S. 1926, The Internal Constitution of the Stars (Cambridge: Cambridge Univ. Press), 115
Fortner, B. F., Lamb, F. K., & Miller, G. S. 1989, Nature, 342, 775
———. 1994, ApJ, submitted
Friedman, J. L., Ipser, J. R., & Parker, L. 1986, ApJ, 304, 115
Ghosh, P., & Lamb, F. K. 1979, ApJ, 232, 259
Graziani, C. 1993, ApJ, 412, 351
Guilbert, P. W., Fabian, A. C., & Rees, M. J. 1983, MNRAS, 205, 593
Hartle, J. B., & Thorne, K. S. 1968, ApJ, 153, 807
Katz, J. I. 1987, High Energy Astrophysics (Menlo Park, CA: Addison-Wesley)
Krolik, J. H. 1988, in Physics of Neutron Stars and Black Holes, ed. Y. Tanaka (Tokyo: Institute for Space and Astronautical Sciences), 21
Lamb, F. K. 1989, in Proc. 23d ESLAB Symp. on Two Topics in X-Ray Astronomy, ed. B. Battrick (ESA SP-296), 215
Lamb, F. K. 1991a, in Frontiers of Stellar Evolution, ed. D. L. Lambert (ASP Conf. Proc. 20), 299
Lamb F. K. 1991b, in Neutron Stars: Theory and Observations, ed. J. Ventura & D. Pines (Dordrecht: Kluwer), 445
Lamb, F. K. 1994, ApJ, submitted
Lewin, W. H. G., van Paradijs, J., & Taam, R. E. 1993, Space Sci. Rev., 62, 223
Mészáros, P. 1992, High-Energy Radiation from Magnetized Neutron Stars (Chicago: Univ. Chicago Press)
Miller, M. C., & Lamb, F. K. 1992, BAAS, 24, 1154
———. 1993, ApJ, 413, L43
———. 1995, in preparation
Misner, C. W., Thorne, K. S., & Wheeler, J. A. 1973, Gravitation (San Francisco: Freeman)
Nemiroff, R. J., Becker, P. A., & Wood, K. S. 1993, ApJ, 406, 590
Park, M.-G., & Miller, G. S. 1991, ApJ, 371, 708
Rees, M. J. 1984, ARA&A, 22, 471
Shapiro, S. L., & Teukolsky, S. A. 1983, Black Holes, White Dwarfs, and Neutron Stars (New York: Wiley-Interscience)
Svensson, R. 1987, MNRAS, 227, 403
———. 1990, in Physical Processes in Hot Cosmic Plasmas, ed. W. Brinkmann, A. C. Fabian, & F. Giovannelli (Dordrecht: Kluwer), 357
van der Klis, M. 1994a, in X-Ray Binaries, ed. W. H. G. Lewin, J. van Paradijs, & E. P. J. van den Heuvel (Cambridge: Cambridge Univ. Press), in press
———. 1994b, in The Lives of Neutron Stars, ed. M. A. Alpar & J. Van Paradijs (Dordrecht: Kluwer), in press
Walker, M. A. 1992, ApJ, 385, 642
Wang, J. C. L., Wasserman, I. M., & Salpeter, E. E. 1988, ApJS, 68, 735
Zdziarski, A. A. 1992, in Testing the AGN Paradigm, ed. S. S. Holt, S. G. Neff, & C. M. Urry (AIP Conf. Proc. 254), 291
Zel'dovich, Ya. B., & Novikov, I. D. 1971, Relativistic Astrophysics (Chicago: Univ. of Chicago Press), 378
Zook, H. A., & Berg, O. E. 1975, Plan. and Sp. Sci. 23, 183