Introduction

The era of direct detection of gravitational waves has commenced. With it comes the opening of a new window onto the universe, which will bring unique insights into black holes, neutron stars, stellar dynamics, and possibly categories or properties of unanticipated sources. In the lectures I will give in this school, I will go through some of the properties of the sources that have been seen with gravitational waves, and some of the currently unresolved questions. Today, we'll start by giving an overview of gravitational radiation. I will also advertise that if you'd like to see a lot more about gravitational radiation and the relevant physics and astrophysics, there is a book I wrote with Nico Yunes of the University of Illinois ("Gravitational Waves in Physics and Astrophysics: An Artisan's Guide", by Miller and Yunes) that was fun to write and which has a lot more details than we're explore in these lectures.

Overview of Gravitational Radiation

As we contemplate the triumphant direct detection of gravitational radiation, it is useful to consider what such detections will teach us about the universe. The first detection, GW150914, was of course of immediate significance because it was a direct confirmation of a dramatic prediction of general relativity: to paraphrase John Wheeler, that spacetime tells sources how to move, and moving sources tell spacetime how to ripple. Now, nearly 100 events have been seen, with many more coming up in future observations.

Beyond the initial detections, gravitational wave science has passed into the realm of astronomy, and is giving us new observational windows onto some of the most dynamic phenomena in the universe. Currently these include merging neutron stars and black holes, and in the future may add supernova explosions and waves from the rotation of lumpy neutron stars, and possibly echoes from the very early history of the universe as a whole. They have also already provided the cleanest tests of predictions of general relativity in the realm of strong gravity, with much more to come.

However, there are important differences from standard astronomy. In electromagnetic observations, in every waveband there are sources so strong that they can be detected even if you know nothing about the source. You don't need to understand nuclear fusion in order to see the Sun! In contrast, as we will see, most of the expected sources of gravitational radiation are so weak that we expect that usually sophisticated statistical techniques will be required to detect them at all (with occasional happy exceptions such as GW150914, which was so strong that it could be seen by eye after moderate bandpass and notch filtering of the data). A standard technique involves matching templates of expected waveforms against the observed data stream. Maximum sensitivity therefore requires a certain understanding of what the sources look like, and thus of the characteristics of those sources. In addition, it will be important to put each detection into an astrophysical context so that the implications of the discoveries are evident.

Before discussing types of sources, though, we need to have some general perspective on how gravitational radiation is generated and how strong it is. We will begin by discussing radiation in a general context.

By definition, a radiation field must be able to carry energy to infinity. If the amplitude of the field a distance r from the source in the direction (θ, ϕ) is $A(r, \theta, \phi)$, the flux through a spherical surface at r, θ, ϕ is $F(r, \theta, \phi) \propto A^2(r, \theta, \phi)$. If for simplicity we assume that the radiation is spherically symmetric, $A(r, \theta, \phi) = A(r)$, this means that the luminosity at a distance r is $L(r) \propto A^2(r)4\pi r^2$. Note, though, that when one expands the static field of a source in moments, the slowest-decreasing moment (the monopole) decreases like $A(r) \propto 1/r^2$, which implies that $F(r) \propto 1/r^4$ and thus $L(r) \propto 1/r^2$, so no energy is carried to infinity. This tells us two things, regardless of the nature of the radiation (e.g., electromagnetic or gravitational). First, radiation requires time variation of the source. Second, the amplitude must scale as $1/r$ far from the source.

We can now explore what types of variation will produce radiation. We'll start with electromagnetic radiation, and expand in moments. Suppose that we are far from some distribution of electric charges, which could be in motion. For a charge density $\rho_e(\mathbf{r})$, the monopole moment is $\int \rho_e(\mathbf{r})d^3r$. We assume that the volume over which we perform the integral encompasses the entire system; no charges can enter or leave. As a result, the monopole moment is simply the total charge Q, which cannot vary, so there is no electromagnetic monopolar radiation. The next static moment is the dipole moment, $\int \rho_e(\mathbf{r})\mathbf{r}d^3r$. There is no applicable conservation law, so electric dipole radiation is possible. One can also look at the variation of currents. The lowest order such variation (the "magnetic dipole") is $\int \rho_e(\mathbf{r})\mathbf{r} \times \mathbf{v}(\mathbf{r})d^3r$. Once again this can vary, so magnetic dipole radiation is possible. The lower order moments will typically dominate the field unless their variation is reduced or eliminated by some special symmetry.

Now consider gravitational radiation. Let the mass-energy density be $\rho(\mathbf{r})$. The monopole moment is $\int \rho(\mathbf{r})d^3r$, which is simply the total mass-energy. This is constant, so there cannot be monopolar gravitational radiation. The static dipole moment is $\int \rho(\mathbf{r})\mathbf{r}d^3r$. This, however, is just the center of mass-energy of the system. In the center of mass frame, therefore, this moment does not change, so there cannot be the equivalent of electric dipolar radiation in this frame (or any other, since the existence of radiation is frame-independent). The counterpart to the magnetic dipolar moment is $\int \rho(\mathbf{r})\mathbf{r} \times \mathbf{v}(\mathbf{r})d^3r$. This, however, is simply the total angular momentum of the system, so its conservation means that there is no magnetic dipolar gravitational radiation either. The next static moment is quadrupolar: $I_{ij} = \int \rho(\mathbf{r})r_i r_j d^3r$. This does not have to be conserved, and thus there can be quadrupolar gravitational radiation. For what it's worth, the actual component that varies is the trace-free quadrupole $\mathcal{I}_{ij} = \int \rho(\mathbf{r}) \left[r_i r_j - \frac{1}{3}\delta_{ij} r^2\right] d^3r$, where δ_{ij} is the Kronecker delta; otherwise, spherical expansion or contraction would lead to time variation and thus gravitational radiation.

This allows us to draw general conclusions about the type of motion that can generate gravitational radiation. A spherically symmetric variation is only monopolar, so it does not produce radiation. No matter how violent an explosion (even a supernova!) or a collapse (even into a black hole!), no gravitational radiation is emitted if spherical symmetry is maintained. In addition, a rotation that preserves axisymmetry (without contraction or expansion) does not generate gravitational radiation because the quadrupolar and higher moments are unaltered. Therefore, for example, a neutron star can rotate arbitrarily rapidly without emitting gravitational radiation as long as it maintains stationarity and axisymmetry and rotates around the axis of symmetry.

This immediately allows us to focus on the most promising types of sources for gravitational wave emission. The general categories are: binaries, continuous wave sources (e.g., rotating stars with nonaxisymmetric lumps), bursts (e.g., asymmetric collapses), and stochastic sources (i.e., individually unresolved sources with random phases; the most interesting of these would be a background of gravitational waves from the early universe).

We can now make some order of magnitude estimates. What is the approximate expression for the dimensionless amplitude h of a metric perturbation, a distance r from a source? Note, by the way, that because gravitational waves are perturbations in spacetime, h is related to the fractional deviation of the spacetime from the Minkowski (flat) spacetime. Thus h, which is also called the strain, is of the order of the fractional change in length induced by a passing gravitational wave, if the length in question is of order the gravitational wavelength.

We argued that the lowest order radiation has to be quadrupolar, and hence depends on the quadrupole moment I. This moment is $\mathcal{I}_{ij} = \int \rho(\mathbf{r}) \left[r_i r_j - \frac{1}{3}\delta_{ij} r^2\right] d^3r$, so it has dimensions MR^2, where M is some mass and R is a characteristic dimension. We also argued that the amplitude is proportional to $1/r$, so we have

$$h \sim MR^2/r \ . \tag{1}$$

We know that h is dimensionless, so how do we determine what else goes in here? In GR we usually set $G = c = 1$, which means that mass, distance, and time all have the same effective "units", but we can't, for example, turn a distance squared into a distance. Our current expression has effective units of distance squared (or mass squared, or time squared). We note that time derivatives have to be involved, since a static system can't emit anything.

Two time derivatives will cancel out the current units, so we now have

$$h \sim \frac{1}{r}\frac{\partial^2(MR^2)}{\partial t^2}\ . \tag{2}$$

Now what? To get back to physical units we have to restore factors of G and c. It is useful to remember certain conversions: for example, if M is a mass, GM/c^2 has units of distance, and GM/c^3 has units of time. Playing with this for a while gives finally

$$h \sim \frac{G}{c^4}\frac{1}{r}\frac{\partial^2(MR^2)}{\partial t^2}\ . \tag{3}$$

Since G is small and c is large, the prefactor is *tiny*! That tells us that unless MR^2 is large, the system is changing fast, and/or r is small, the metric perturbation is minuscule.

Let's make a very rough estimate for a circular binary. Suppose the total mass is $M = m_1 + m_2$, the reduced mass is $\mu = m_1 m_2/M$, and the semimajor axis is a, so the orbital frequency Ω is given by $\Omega^2 a^3 = GM$. Without worrying about precise factors, we say that $\partial^2/\partial t^2 \sim \Omega^2$ and $MR^2 \sim \mu a^2$, so

$$h \sim (G^2/c^4)(\mu/r)(M/a)\ . \tag{4}$$

This can also be written in terms of orbital periods, and with the correct factors put in we get, for example, for an equal-mass system

$$h \approx 10^{-22}\left(\frac{M}{2.8\,M_\odot}\right)^{5/3}\left(\frac{0.01\ \mathrm{sec}}{P}\right)^{2/3}\left(\frac{100\ \mathrm{Mpc}}{r}\right)\ , \tag{5}$$

which is scaled to a double neutron star system. This is really, really, small: it corresponds to less than the radius of an atomic nucleus over a baseline the size of the Earth. That's why it is so challenging to detect these systems!

Remarkably, though, the flux of energy is *not* tiny. To see this, let's calculate the flux given some dimensionless amplitude h. The flux has to be proportional to the square of the amplitude and also the square of the frequency f: $F \sim h^2 f^2$. This currently has units of frequency squared, but the physical units of flux are energy per time per area. Replacing factors of G and c, we find that the flux is

$$F \sim (c^3/G)h^2 f^2\ . \tag{6}$$

Now the prefactor is *enormous*! For the double neutron star system above, with $h \sim 10^{-22}$ and $f \sim 100$ Hz, this gives a flux of a few hundredths of an erg cm^{-2} s^{-1}. For comparison, the flux from Sirius, the brightest star in the night sky, is about 10^{-4} erg cm^{-2} s^{-1}! That means that if you could somehow absorb gravitational radiation perfectly with your eyes, you would see hundreds to thousands of events per year brighter than every star except the Sun. To put it another way, the energy per time emitted by a comparable-mass double black hole coalescence, during the last part of its merger, is tens of times greater than the energy per time emitted by every star in the visible universe *combined* during that same time (!!!). What this really implies, of course, is that gravitational radiation interacts *very* weakly with matter, which again means that it is mighty challenging to detect.

Going back to the amplitude $h \sim (\mu/r)(M/a)$, we can rewrite the amplitude using $f \sim (M/a^3)^{1/2}$, to read

$$\begin{aligned} h &\sim \mu M^{2/3} f^{2/3}/r \\ &\sim M_{\rm ch}^{5/3} f^{2/3}/r \end{aligned} \tag{7}$$

where $M_{\rm ch}$ is the "chirp mass", defined by $M_{\rm ch}^{5/3} = \mu M^{2/3}$ and thus $M_{\rm ch} = \mu^{3/5} M^{2/5}$. The chirp mass is named that because it is this combination of μ and M that determines how fast the binary sweeps, or chirps, through a frequency band. When the constants are put in, the dimensionless gravitational wave strain amplitude (which, remember, is roughly the fractional amount by which a separation changes as a wave goes by) measured a distance r from a circular binary of masses M and m with a binary orbital frequency $f_{\rm bin}$ is (Schutz 1997)

$$h = 2(4\pi)^{1/3} \frac{G^{5/3}}{c^4} f_{\rm GW}^{2/3} M_{\rm ch}^{5/3} \frac{1}{r} \ , \tag{8}$$

where $f_{\rm GW}$ is the gravitational wave frequency. Redshifts have not been included in this formula.

The luminosity in gravitational radiation is then

$$\begin{aligned} L &\sim 4\pi r^2 f^2 h^2 \\ &\sim M_{\rm ch}^{10/3} f^{10/3} \\ &\sim \mu^2 M^3/R^5 \ . \end{aligned} \tag{9}$$

The total energy of a circular binary of radius R is $E_{\rm tot} = -G\mu M/(2R)$, so we have

$$\begin{aligned} dE/dt &\sim \mu^2 M^3/R^5 \\ \mu M/(2R^2)(dR/dt) &\sim \mu^2 M^3/R^5 \\ dR/dt &\sim \mu M^2/R^3 \ . \end{aligned} \tag{10}$$

We see that gravitational radiation is emitted much more strongly when the orbital radius is small. If we take an intuitive leap, this would suggest that if the orbit is eccentric, more radiation will be emitted at pericenter than at apocenter. This would therefore have the character of an impulsive force. With such a force, the orbit will return to where the impulse was imparted. That means that the pericenter distance will remain roughly constant, while the energy losses decrease the apocenter distance. As a consequence, the eccentricity decreases. In general, gravitational radiation will decrease the eccentricity of an orbit.

The detailed formulae bear this out, as derived by Peters and Matthews (1963) and Peters (1964). If the orbit has semimajor axis a and eccentricity e, their lowest-order rates of change of the orbital parameters are

$$\left\langle \frac{da}{dt} \right\rangle = -\frac{64}{5} \frac{G^3 \mu M^2}{c^5 a^3 (1-e^2)^{7/2}} \left(1 + \frac{73}{24} e^2 + \frac{37}{96} e^4\right) \tag{11}$$

and

$$\left\langle \frac{de}{dt} \right\rangle = -\frac{304}{15} e \frac{G^3 \mu M^2}{c^5 a^4 (1-e^2)^{5/2}} \left(1 + \frac{121}{304} e^2\right) \,, \tag{12}$$

where the angle brackets indicate an average over an orbit. For an orbit of initial semimajor axis a_0 and eccentricity e_0, to within $\sim 30\%$ the inspiral time to merger is

$$T \approx 4 \times 10^{17}\ \mathrm{yr} \left(\frac{M_\odot^3}{\mu M^2}\right) \left(\frac{a_0}{1\ \mathrm{au}}\right)^4 \left(1 - e_0^2\right)^{7/2} \,, \tag{13}$$

where "au"="astronomical unit" (the average Earth-Sun distance, which is about 1.5×10^{13} cm). This means that a circular binary of two $\sim 10\ M_\odot$ black holes needs to start with $a_0 \lesssim 0.1$ au to merge in the $\sim 10^{10}$ yr age of the universe.

One can show that these rates imply that the quantity

$$a e^{-12/19} (1 - e^2) \left(1 + \frac{121}{304} e^2\right)^{-870/2299} \tag{14}$$

is constant throughout the inspiral. If we ignore the final factor (which is always between 0.88 and 1), we can write this as $a(1-e)(1+e)e^{-12/19} \approx$ const. For high eccentricities such that $1-e \ll 1$, $1+e$ and $e^{-12/19}$ are roughly constant, so $a(1-e) = r_p \approx$ const, which means that the pericenter distance r_p is roughly constant as promised. For low eccentricities such that $1 - e^2 \approx 1$, we get $ae^{-12/19} \approx$ const. The orbital frequency (which is half the dominant gravitational wave frequency when $e \ll 1$) is $f \propto a^{-3/2}$, which means that $f \propto e^{-18/19}$, or roughly $e \propto f^{-1}$. Thus for low eccentricities, the eccentricity roughly scales as the reciprocal of the frequency. This means that binary sources at the high frequencies $f \gtrsim 20$ Hz detectable using ground-based instruments can usually be considered to be effectively circular.

Let us conclude with an idea of the frequency range available for a given type of binary. There is obviously no practical lower frequency limit (just increase the semimajor axis as much as you want), but there is a strict upper limit. The two objects in the binary clearly won't produce a signal higher than the frequency at which they touch. If we consider an object of mass M and radius R, the orbital frequency at its surface is $\sim \sqrt{GM/R^3}$. Noting that $M/R^3 \sim \rho$, we can say that the maximum frequency involving an object of density ρ is $f_{\max} \sim (G\rho)^{1/2}$. This is actually more general than just orbital frequencies. For example, a gravitationally bound object can't rotate faster than that, because it would fly apart. In addition, you can convince yourself that the frequency of a sound wave that involves most of the object can't be greater than $\sim (G\rho)^{1/2}$. Therefore, $\sim (G\rho)^{1/2}$ is a general upper bound on dynamical frequencies.

This tells us, therefore, that binaries involving main sequence stars can't have frequencies greater than $\sim 10^{-3} - 10^{-6}$ Hz, depending on mass, that binaries involving white dwarfs can't have frequencies greater than $\sim 0.1 -$ few Hz, also depending on mass, that for neutron stars the upper limit is $\sim 1000 - 2000$ Hz, and that for black holes the limit depends inversely on mass (and also spin and orientation of the binary). In particular, for black holes the maximum imaginable frequency is on the order of $10^4(M_\odot/M)$ Hz at the event horizon, but in reality the orbit becomes unstable at lower frequencies (more on that in the next lecture).

The net result is that for ground-based interferometers such as LIGO, Virgo, KAGRA (collectively LVK), as well as GEO-600 and the future LIGO-India, which are sensitive to frequencies $\sim 20 - 2000$ Hz, the only individual sources that will be detected are neutron stars and black holes and their creation events (supernovae); some might argue that cuspy cosmic strings might fall into this category, but we'll leave that for a later discussion. In the next two lectures we will therefore lay out the astrophysical basics of black holes and neutron stars, and will thrown in white dwarfs for good measure.