Problem set 5

1. In this problem you will derive the interaction cross section including gravitational focusing. You will start by assuming that a test particle of some very small mass starts at effectively infinite distance from a compact body of mass M. The initial speed of the test particle is v with respect to M, and if it traveled in a straight line then the closest it would get to M is a distance b; b is called the impact parameter of the trajectory. As a result, the specific angular momentum of the trajectory relative to M is bv and the specific energy of the trajectory is $\frac{1}{2}v^2$. By conserving energy and angular momentum, determine the value of b such that the closest approach to M is r_p. **Hint:** at the closest approach, the velocity vector is perpendicular to the direction to M. The effective cross section for an interaction of closest approach r_p or closer is then $\Sigma = \pi b^2$.

2. In this problem we will perform calculations related to the production of a binary by direct capture during a two-body encounter. The scenario is that in a dense stellar system two black holes, which are initially unbound with respect to each other, pass close enough to each other that the gravitational radiation released during the encounter binds the black holes into a binary. Because the relative speed at great distances (typically $\sim$tens of km s^{-1}) is tiny compared with the speed at pericenter in such encounters (typically tens of thousands of km s^{-1}), we can approximate the orbit as parabolic. The energy release in gravitational waves for a parabolic encounter between two masses m_1 and m_2 with closest approach r_p is

$$\Delta E = \frac{85\pi G^{7/2} m_1^2 m_2^2 (m_1+m_2)^{1/2}}{12\sqrt{2} c^5 r_p^{7/2}} \; . \tag{1}$$

Given this:

1. For an initial relative speed at a large distance of v_∞, calculate the closest approach r_p such that ΔE is equal to $\frac{1}{2}\mu v_\infty^2$, where $\mu = m_1 m_2/(m_1+m_2)$ is the reduced mass. Thus encounters with closest approach distances of r_p or smaller will result in a bound binary.
2. Calculate the effective cross section of such encounters, assuming that at r_p the relative speed is much larger than v_∞. **Hint:** gravitational focusing is dominant in this limit.
3. Suppose that the core of a globular cluster has $v_\infty = 10$ km s^{-1}, and 100 black holes each with mass 10 $M_\odot$, at a number density of 10^5 pc^{-3}. Given the cross section that you found in part b, compute the expected number of double black hole mergers you would expect in the globular in 10^{10} years. Here the assumption (which you should check) is that once black holes are captured into a binary by this mechanism, coalescence is rapid.

3. Dr. Sane doesn't understand all this focus on binary compact object mergers. Instead, direct collisions of single neutron stars in clusters with each other will make wonderful burst sources. Dr. Sane has requested that you work out the numbers. Suppose that you consider a dense globular cluster, such that in the center the number density of neutron stars is 10^6 pc^{-3} and there are 1000 total neutron stars per cluster. Suppose that each neutron star has a radius of 10 km and mass of $1.5\, M_\odot = 3 \times 10^{33}$ g, and that the typical random speed in the cluster is 10 km s^{-1}. To within an order of magnitude, calculate how often two neutron stars in a given cluster will hit each other. If there are 10^{10} such clusters in the universe, how often will this happen in the universe? **Hint:** be careful when you calculate the cross section for collisions, because gravitational focusing is important.

4. If you have a cluster of stars moving around and interacting gravitationally, then an important concept in dynamics is the *relaxation time*. This, essentially, is the time needed for a given object (say, a star) to double or halve its semimajor axis; that is, it's the time needed for the semimajor axis to change substantially. The local relaxation time is

$$t_{\rm rlx}(r) = \frac{1}{\ln\Lambda}\frac{\sigma^3(r)}{G^2M^2n(r)} \ , \tag{2}$$

where $\ln\Lambda \sim 10$ comes from the "Coulomb integral" (which is a factor that lumps together various detailed effects), $\sigma(r)$ is the *local* velocity dispersion, M is the typical mass of an object, and $n(r)$ is the local number density of objects. Consider a region $r < r_{\rm infl}$, where $\sigma(r)$ is given by the Keplerian orbital speed.

1. If $n(r) \propto r^{-3/2}$ (a typical profile), how does the relaxation time depend on r?
2. In contrast, for $r \gg r_{\rm infl}$, assume that $n(r) \propto r^{-2}$ and $\sigma(r)$ is constant. Then how does the relaxation time depend on r?

5. One of the ways that black holes can acquire mass, and possibly grow into supermassive black holes, is Bondi-Hoyle-Lyttleton accretion. In this process, gas that moves at a speed v relative to the black hole (we assume here that v is much larger than the sound speed of the gas) is gravitationally deflected by the hole, heats itself, shocks as a result, releases energy, and if it is close enough to the hole it is then bound and eventually accretes into the hole. For a black hole of mass M, the cross section for this type of accretion is $\Sigma_{\rm BHL} = \pi(GM/v^2)^2$ times a numerical factor close to unity that depends on the details of the flow.

But what if the matter does not interact with itself? An example would be dark matter. Then, for the matter to accrete it needs to hit the hole directly. For a nonrotating black hole (rotating black holes have slightly different numbers), capture requires that the angular momentum per unit mass is less than $4GM/c$. Given this, compute the ratio of the cross section for direct impact accretion to the cross section for Bondi-Hoyle-Lyttleton accretion,

and comment on the implications for rapid growth of black holes by accretion of dark matter.
Hint: assume that at a great distance from the black hole, the dark matter is moving at the same speed v relative to the hole as is the gas and note that in galaxies v is typically a few hundred kilometers per second.