Problem set 6

1. Neutron stars could range over roughly a factor of 2 in mass (say, $1.1 - 2.2\ M_\odot$). Given that, and assuming that you effectively measure the chirp mass perfectly from a NS−NS binary but have no direct measurement of the mass ratio, how precisely do you know the total mass of the binary? This can be important if, for example, you find clear evidence that the result of the merger was prompt black hole formation, because then you know an upper limit to the maximum mass of a neutron star.

2. We recall from the first lecture that the inspiral time for a binary of total mass $M = m_1+m_2$ and reduced mass $\mu = m_1 m_2/\mu$, with initial semimajor axis a_0 and initial eccentricity e_0, is

$$T \approx 4 \times 10^{17}\ \text{yr} \left(\frac{M_\odot^3}{\mu M^2}\right)\left(\frac{a_0}{1\ \text{au}}\right)^4 \left(1-e_0^2\right)^{7/2}\ . \tag{1}$$

Your tasks:

a. For an initially quasicircular orbit $e_0 = 0$, re-express this in terms of gravitational wave frequency $f_{\rm GW} = 2f_{\rm orb} = (1/\pi)(GM/a^3)^{1/2}$, normalizing the frequency to 20 Hz, which is roughly the lower limit of ground-based instruments such as LVK.

b. Given your answer to the previous question, how long would it take for a $10\ M_\odot - 10\ M_\odot$ black hole binary to spiral from 20 Hz to merger (at much larger than 20 Hz)? What about for a $1.4\ M_\odot - 1.4\ M_\odot$ neutron star binary?

c. Assume that the volume rate of BH-BH coalescences (assumed to be $10\ M_\odot - 10\ M_\odot$) is 40 $\text{Gpc}^{-3}\ \text{yr}^{-1}$ and that the volume rate of NS-NS coalescences (assumed to be $1.4\ M_\odot - 1.4\ M_\odot$) is 400 $\text{Gpc}^{-3}\ \text{yr}^{-1}$. Assume that the stochastic background (which we recall is the sum of individually undetected or unresolved sources) from each of the two categories of sources comes from redshifts less than $z = 1$, i.e., comoving radial distances less than 3.4 Gpc. How does the typical time between coalescences between black holes compare with the duration of the black hole signal above $f_{\rm GW} = 20$ Hz? What about for neutron stars? What does this imply about the expected differences between the nature of the stochastic background produced by neutron stars, from the stochastic background produced by black holes?

d. In the previous part we extended the integration to $z = 1$. But an argument similar to Olbers' Paradox suggests that the farther we go, the larger the background (indeed, the total integrated flux is proportional to the distance). If the rate of coalescences is proportional to the star formation rate then we'd expect an even bigger boost from high redshifts. Is there a reason that higher redshifts might *not* contribute the dominant number of events?

3. Here's a question for which there is currently no consensus answer, so you can apply your

creativity. The black holes in the gravitational wave events thus far observed appear to have low spin. However, the black holes observed electromagnetically in our Galaxy have a broad range of spin parameters, more or less from 0 to 1. Taking into account that (a) via either method we see only a tiny, tiny fraction of all stellar-mass black holes, and (b) the observing techniques and thus (inevitably) selection effects are different from each other, why might it be that the spin distributions are different?

Challenge: Make your own estimate of the rate per volume of BH-BH mergers (expressed in number per Gpc^3 per year), including the 90% credible interval, based on the three events reported in the O1 run. The first Advanced LIGO run had 49 total days in which both detectors were taking data, so that will be our baseline time. Potentially relevant numbers are: GW150914 was at a distance of 420 Mpc (we'll ignore the uncertainties for simplicity) and had a signal to noise ratio of 23.7; GW151226 was at a distance of 440 Mpc and had a signal to noise ratio of 13.0; LVT151012 was at a distance of 1 Gpc and had a signal to noise ratio of 9.7. Suppose that the threshold for announcing a detection is a signal to noise ratio of 12.0 (note that LVT151012 was a marginal detection), and remember that for a given event the distance scales as the reciprocal of the signal to noise ratio.

a) With no other information, what would be your best estimate for the rate per volume based on each of the events individually (i.e., without combining them or estimating uncertainties)?

b) How should you estimate the uncertainties for each event individually? More specifically, how would you calculate the 90% credible interval for the rate based on each event individually?

c) How should you combine the information from the three events? Do this without, then with, the uncertainties included.

d) Suppose now that you are given the information that one of the events (pick any of them) was in a direction to which Advanced LIGO was unusually sensitive. What effect, if any, would this have on your best estimate of the rate based on that event (i.e., would it decrease your best estimate, increase your best estimate, or leave it unchanged)?

e) Same question as d), but with regard to the orientation: suppose that one of the events was known to have its binary orbital axis pointed nearly towards us, which means that we see a high amplitude compared to the orientation-averaged amplitude. What effect would this have on your best estimate of the rate from that event alone?