**Equations of Stellar Structure**

We're getting closer to being able to construct simple models of stars. To do this we need to write two more equations relating various quantities; an energy transfer equation, which relates the gradient of the energy density to the luminosity, and an equation of state to relate the pressure to the density and temperature. Once we have that, we have a complete set of equations (in principle) and will apply them in a simple model. In the next several classes we will then focus on particular aspects of these relations: energy generation, equations of state, and opacities.

For energy transfer, we'd like to think about the *net flux* of energy a given location. Since we're restricting things to spherical symmetry, the only direction in which such flux can be nonzero is the radial direction. In the interior of stars (which is what we care about), the mean free path is small enough that we can think about diffusion (i.e., particles scatter or interact an enormous number of times before they leave the star). Thus, the flux depends in some way on the energy density. **Ask class:** if the energy density is constant throughout the star, will there be net flux? No, because as much goes down as up. So, **Ask class:** how must the flux be related to the energy density? Through the gradient of the energy density, which is $dE/dr$ since spherical symmetry applies. Now, let's say that the energy is transported by some particle with a mean free path $\ell$. **Ask class:** for all else equal, will the net flux be larger if $\ell$ is larger or smaller? Larger, because a larger change in energy density is sampled. Conversely, if the opacity $\kappa$ is large (meaning that the mean free path is small), the net flux must be small. Therefore, $F \sim \kappa^{-1} dE/dr$. **Ask class:** if $r$ and $F$ are both positive in the outward direction, is the sign positive or negative in the equation for $F$? It must be negative, because for outward flow of energy we need $E$ to be larger at smaller $r$, and thus we need $dE/dr$ to be negative.

Let us consider first transport of energy by radiation. Other possibilities exist; for example, in parts of the Sun and other stars convection is the primary way that energy is transported, and in degenerate stars it is conduction. But it's instructive to start with radiation transport, and this is the dominant mode for substantial fractions of the volumes of many stars.

The energy density of radiation is $aT^4$, and when various other factors are put in (see our textbook) the flux is $F_r = -(c/3\kappa\rho)d(aT^4)/dr$, so that the luminosity is

$$L_r = -\frac{4\pi r^2 c}{3\kappa\rho}\frac{d(aT^4)}{dr}\ . \tag{1}$$

This may also be written in the Lagrangian form (i.e., with $dM_r$ and the fundamental differential instead of $dr$) using the relation $dM_r = 4\pi r^2 \rho dr$.

The opacity has dimensions of cm$^2$ g$^{-1}$, and is therefore the total cross section of a given mass of material. We'll talk about the opacity in much greater detail later, but for now note that $1/(\kappa\rho)$ has dimensions of centimeters, and is in fact the mean free path. The factor of $c$ in equation 1 is in the numerator because radiation travels at the speed of light; clearly, the faster your energy-carrying particles move, the more flux is transported, for all else equal.

The full calculation of opacities can be awful. Luckily, for many applications one can write it as

$$\kappa = \kappa_0 \rho^n T^{-s} \ , \tag{2}$$

where $\kappa_0$, $n$, and $s$ are constants. For example, for Thomson scattering (recall that this is scattering off of a free electron of a photon with an energy much less than the electron rest-mass energy $m_e c^2$), $n = s = 0$, whereas for a Kramer's law of opacity $n = 1$ and $s = 3.5$, which is useful for various atomic opacities. The final equation we'd like is the equation of state, which relates pressure to density and temperature. In the same spirit as the opacity, we can write

$$P = P_0 \rho^{\chi_\rho} T^{\chi_T} \ , \tag{3}$$

with $P_0$, $\chi_\rho$, and $\chi_T$ constant. For example, for an ideal gas $P = nkT$ so $\chi_\rho = \chi_T = 1$. Similarly, the energy generation rate $\epsilon$ (such that $dL/dM = \epsilon$) is often written as $\epsilon = \epsilon_0 \rho^\lambda T^\nu$. Note that all these power-law forms are approximations (which are usually only applicable over some range in density, temperature, etc., but they can be surprisingly good for many applications), and that they also really depend on things such as the composition.

We can put these together in our four fundamental equations of stellar structure:

$$\begin{array}{ccc}
 & \text{Eulerian} & \text{Lagrangian} \\
\text{Mass equation} & M(r<r_0) = \int_0^{r_0} \rho 4\pi r^2\, dr & M(r<r_0) = \int_0^{r_0} \rho 4\pi r^2\, dr \\
\text{Hydrostatic equation} & dP/dr = -\rho GM/r^2 & dP/dM = -GM/4\pi r^4 \\
\text{Energy balance} & dL/dr = 4\pi r^2 \rho\epsilon & dL/dM = \epsilon \\
\text{Energy transfer} & L = -\frac{4\pi r^2 c}{3\kappa\rho}\frac{d(aT^4)}{dr} & L = -\frac{(4\pi r^2)^2 c}{3\kappa}\frac{d(aT^4)}{dM}
\end{array} \tag{4}$$

**Dimensional Analysis and Homology Relations**

After considerable thought, I have decided that I do not want to go into this subject at the depth that our textbook covers. The reason is that (see below) the assumptions in this analysis are not correct in detail and indeed are importantly incorrect in some ways, so I don't want to focus on it too much. On the other hand, dimensional analysis is an important way to get physical insight, so I also don't want to skip the subject entirely. As a compromise, I'll give you the basic principles and then stop.

The idea of homology relations is to suppose that there is a single "standard star" and that all other stars are scaled versions of this. Among other things this means that we assume spherical models with the same uniform composition and microphysics. **Note:** this means that there is no preferred scale, and thus that in using these relations we are assuming that all stars are intrinsically alike. This implies that various functions are power laws, since these have no scale either (true *only* of pure power laws, in the sense that, e.g., if $f(ax)/f(x)$ is independent of $x$ for any constant $a$, $f$ must be a pure power law). A familiar example is the Newtonian orbit of two bodies: the two bodies each orbit the center of mass of the system, each with the same eccentricity, and the ratio of the semimajor axes is the inverse of the ratio of the masses. You couldn't tell by looking at only the shape of the orbits (rather than their size, or the orbital period) what the masses are or what the separation is. If you go to general relativity, however, then there are some scales that do enter, although for black holes, again, changing the mass does not fundamentally alter the nature of the spacetime exterior to the black hole.

The lesson is that for scale-free quantities, power laws rule! Whenever there is a preferred scale, the power laws are broken. We'll keep this in mind because for homology relations we will assume power laws, but they are valid only as long as the scale free property works (e.g., the microphysics is the same for all the stars in our sample).

Thus, we will assume that $r$ and $M_r$ in a star of radius $R$ and mass $M$ are related to each other, with respect to a reference star of radius $R_0$ and mass $M_0$: if

$$r = \frac{R}{R_0} r_0 \tag{5}$$

then

$$M_r = \frac{M}{M_0} M_{r0,0} \ . \tag{6}$$

That is, the mass interior to $r$ in the general star is equal to $M/M_0$ times the mass interior to $r_0$ in the reference star.

In a homology treatment, we then assume that various quantities are related to the total mass via power laws. For example, we assume that $R \propto M^{\alpha_R}$ and $\rho \propto M^{\alpha_\rho}$. From that and the mass equation $\rho R^3 \propto M$, we get (by taking the derivative of the log of that relation) $\alpha_\rho + 3\alpha_R = 1$. We also assume that, say, the pressure (and opacity, and energy generation rate, and ...) can also be written with power laws. For example, we assume an equation of state $P = P_0 \rho^{\chi_\rho} T^{\chi_T}$. From that (see book for details), you can get various relations between the exponents (i.e., the $\alpha$s and the $\chi$s) using particular assumptions.

This is a lot of assumptions; how well does it do in practice? As one example, we can consider moderately massive main sequence stars, for which the opacity is dominated by electron scattering, the equation of state is close to that of an ideal gas, and hydrogen fuses to helium mainly through the CNO cycle (more on this later in the course). Then when you go through everything you get $(R/R_\odot) = (M/M_\odot)^{0.78}$ and $(L/L_\odot) = (M/M_\odot)^3$. Empirically, and also using full numerical models, the exponents are more like 0.78 and 3.5, so our homology relation didn't do too badly. Using the empirical relation and assuming that during the main sequence stars convert a nearly fixed fraction (about 15%) of their hydrogen to helium, we can then estimate the main sequence lifetime. The total mass available to fuse is $M$, so the total energy available is also proportional to $M$ and the time is thus proportional to $M/L$, so $T = T_\odot[(M/M_\odot)/(L/L_\odot)]$, or $T \approx 10^{10}(M/M_\odot)^{-2.5}$ yr.

How can we assess the utility of homology relations and dimensional analysis in this context? This approach was essential decades ago, when computers were terrible. Even now, we can get insights in a simplified treatment such as this that we do not get when we just throw everything at someone else's sophisticated code and assume that the results are correct. I think the right approach is to understand the *principle* of dimensional analysis, and also to think carefully about the assumptions that go into any particular analysis so that we can have a reasonable sense for when those assumptions fail.

With that in mind, even though we haven't encountered details yet, **Ask class:** let's brainstorm about ways that our assumptions above might fail for sufficiently low-mass or high-mass stars. Even if we don't know, it's good to think about this in advance so that we are better prepared when we go into the relevant details.

## Observables of Stars, Conventions

For a given star, we can directly observe only the flux (and in some cases the polarization) as a function of photon wavelength. For close enough stars, and using (recently) the amazing power of the Gaia mission, parallax gives us a way to estimate their distance. This means that we can use the observed flux plus the distance to estimate the intrinsic luminosity as a function of wavelength, although we do have to worry about wavelength-dependent absorption of light by the interstellar medium, and to a much lesser degree about the fact that when a star rotates it does not emit perfectly isotropically, so our line of sight might not be perfectly representative.

We'd like to quantify the flux at each wavelength. If we did this now we would do so by asking about the energy per area per time per interval of wavelength, at that wavelength. We could instead do this by frequency rather than wavelength given that they are interchangeable. Indeed, this is more or less how people quantify flux in radio waves, X-rays, and gamma-rays.

But because optical astronomy started far earlier than at any other wavelength (since we can just use our eyes!), we are stuck in the optical and in much of the infrared and ultraviolet bands with a system originated by the Greek astronomer Hipparchus more than two millennia ago. Thus in these bands we are saddled with the curse of magnitudes. This, as you probably know, is a logarithmic system: 5 magnitudes equals a factor of 100 in flux in a given band. If a star is *dimmer*, its magnitude is *greater*. The historical antecedent is probably that people would talk about "a star of the first magnitude" being brighter than one of the second magnitude, and probably the proportions between magnitudes are about the level distinguishable by eye. At a dark site, with good vision, magnitude 6 is the typical naked eye limit. The brightest star in the night sky is Sirius, at magnitude -1.5.

All of that is the *apparent* magnitude, which is basically how bright the star appears to be. But we can also define the *absolute* magnitude, which is the magnitude that star would appear to have if it were 10 pc from us (remember that a parsec is about 3.26 light years and is the distance such that the parallax from 1 au of motion is one arcsecond; thus *par*ralax arc*sec*ond). The Sun has an absolute V magnitude of 4.8 (why "V"? See below for the horrifying truth). This is one way of determining the distance of an object, because the distance follows if we know the true and the apparent brightness. However, absorption by intervening dust and gas can make such calculations more difficult.

As some of you might know, it's actually worse than that. Our first thought regarding the flux of stars would probably be that we want to sum up all of the light from a star, and then use the magnitude system from that point. But since we want to know the flux *as a function of wavelength*, we need some system to do that. A logical approach is to do so wavelength range by wavelength range; ideally we'd choose really tiny wavelength ranges to maximize our information, but for faint things that gives us too little flux. Therefore, instead, we need to define different, and at least somewhat broad, bands in the spectrum that we integrate to get the flux in that band. This is called photometry, in distinction to spectroscopy (which involves breaking the spectrum into very narrow intervals by using, e.g., a diffraction grating or something similar).

But the reason that I'm moaning about photometry is that *there isn't a single unique system*! In practice (at least in the olden days) the way that you get photometric information is by applying filters to the light that comes in a telescope. Those filters let in only some range of wavelengths, and are typically designed to avoid, e.g., strong spectral lines from the Earth's atmosphere. But historically, and even now to some degree, the exact nature of those filters depends on different choices. For example, you could choose the Johnson system or the SDSS system. Luckily, there are websites where you can convert between the different magnitudes, but you have to be careful.

The next bit of terminology to recall is spectral types. You may remember that the spectral sequence is OBAFGKM (and sometimes people add RNS to the end). Don't worry so much about details, but the general idea is that the "earlier" (towards O) the type is the more massive and hotter it is, and the "later" (towards M) the less massive and cooler it is. This earlier/later nomenclature is a relic of the idea that this actually represented an evolutionary sequence, but in fact these are entirely separate stars. There are also subdivisions within this: for a given spectral class (say, G, the spectral class of the Sun), there are attached numbers with a full range of 0 through 9; more massive/hotter/luminous stars have lower numbers. The Sun is a G2. Finally, for a given spectral type (which depends on the spectrum rather than on anything else), there are luminosity classes: I is most luminous, VI is least luminous. The Sun, at luminosity class V, is technically a "dwarf". No, it doesn't make sense to me either, but if you work in this field you have to learn what things are called.

Something that *does* make a bit of physical sense is the Hertzsprung-Russell, or H-R, diagram. The observers's version of the H-R diagram has absolute magnitude on the vertical axis and "color" which is the difference between magnitudes in different bands (which, since magnitudes are logarithmic, is the log of the ratio of fluxes in different bands) on the horizontal axis. The theorist's version has luminosity on the vertical axis and temperature on the horizontal axis (oddly, increasing toward the left). The difference is that the observer's version relates to directly observable quantities, whereas the theorist's version plots more physically meaningful quantities.

As you know, the H-R diagram has proven to be extraordinarily useful in categorizing stars. You can see a well-populated band from upper left to lower right: the main sequence. Up and to the right from the main sequence (i.e., cooler but more luminous) you have the red giants. Down and to the left (i.e., hotter but less luminous), you have the white dwarfs. All in all, with enough stars you can see the evolutionary sequences of stars, determine which are common vs. rare, and determine the lifetimes of stars of different masses given observations of star clusters of different ages. You can even distinguish the presence of binaries, which (if both stars are the same) have the same color but are twice as luminous as a single star. It's an impressive tool. Of course you have to be a little careful, e.g., dust absorption reduces the flux and makes the color redder, but this was a great success in categorizing stars.