Stellar energy sources: gravitation

In the first class we had fun thinking about various energy sources that might power the Sun. What eliminates all possibilities other than nuclear fusion is the combination of (1) the energy per mass emitted in the Sun's lifetime, and (2) the stability of the energy source (the Sun has been going strong for more than 4 billion years).

In more detail, if we multiply the Sun's luminosity (about 3.8×10^{33} erg s^{-1}) by its lifetime (about 4.6 billion years), we get a total of 5.5×10^{50} erg. Since the Sun's mass is about $M_\odot = 2 \times 10^{33}$ g, this means that the energy release over the Sun's lifetime up until now is 2.8×10^{17} erg g^{-1}. If the Sun were to convert all of its mass into energy, then the energy per mass would be c^2, or about 9×10^{20} erg g^{-1}. Chemical processes (burning of paper or gasoline, or energy used by hamsters on a wheel!) generate roughly one eV (typical energy of electrons in an atom) per proton mass, or $\sim 1.6 \times 10^{-12}$ erg per $\sim 1.7 \times 10^{-24}$ g, i.e., about 10^{12} erg g^{-1}, which is orders of magnitude short of what is needed. For gravity, and remembering that the gravitational energy of mass m at a distance R from mass M is GMm/R, the energy per mass is $GM_\odot/R_\odot$ where $R_\odot \approx 7 \times 10^{10}$ cm, and is thus about 1.9×10^{15} erg g^{-1}, which is also too little (note: since the Sun is centrally concentrated the actual energy release is greater, but it still falls short by a lot). Nuclear fission has enough energy per mass (maybe 10^{18} erg g^{-1}), but above a critical mass fission happens explosively, so it can't sustain itself for billions of years.

So nuclear fusion is it for the Sun, and in the following two lectures we'll go into fusion in detail. But here we will focus on gravity as an energy source. The reason we'll do that is because it is conceptually simpler than fusion, and because gravity actually is the main source of energy for a number of interesting astronomical objects such as protostars, giant planets, and accreting compact objects.

To begin, let's be a little more careful about how we estimate the energy released in assembling a spherical ball of gas (which to first order is a star or a gas giant planet or a protostar!). We will think about this in terms of *binding energy.* Basically, to figure out how much gravitational energy is released in the assembly of our spherical ball, we calculate how much energy it would take to strip it away layer by layer and take each layer to infinity. Say that the outer layer has a mass dM, and is at radius R outside of the rest of the star, which has mass M. Then the gravitational binding energy is $GMdM/R$; note that this is positive, because it's the energy we would need to put into the layer dM to get it to infinity. Because by assumption the star is spherically symmetric, the density is just a function of r, $\rho = \rho(r)$, and thus $dM = \rho(r) 4\pi r^2 dr$ at radius r. Having removed that layer, we can now remove the next one down, and so on. Overall, the total gravitational binding energy is then

$$E_{\rm grav} = \int_0^R \frac{GM(<r)}{r}\rho(r)4\pi r^2 dr\ , \tag{1}$$

where $M(<r)$ is the mass interior to r. For example, suppose that $\rho(r)$ is constant, $\rho(r)=\rho_0$. Then $M(<r)=\rho_0(4/3)\pi r^3$ and thus

$$\begin{aligned} E_{\rm grav} &= \int_0^R G\rho_0(4/3)\pi r^2\rho_0 4\pi r^2 dr \\ &= \int_0^R G\rho_0^2(16/3)\pi^2 r^4 dr \\ &= (16/15)\pi^2\rho_0^2 R^5 \\ &= (9/15)G[\rho_0(4/3)\pi R^3]^2/R \\ &= (3/5)GM^2/R\ . \end{aligned} \tag{2}$$

Dimensionally, the energy has to be proportional to GM^2/R; the only question is the prefactor. For a constant-density star it is 3/5. Stars are centrally concentrated, because the central layers are squeezed together by the weight of the layers above. Therefore, the prefactor is actually larger than 0.6, and by how much depends on the structure of the star. For example, the references I have found suggest that the effective prefactor for the Sun is very close to 1, but for a red giant (large tenuous envelope and a very centrally concentrated core) the prefactor would be considerably larger. But for the most part you're basically okay with $E_{\rm grav}\approx GM^2/R$.

So now let's see what this means for the Sun. $GM_\odot^2/R_\odot = 3.8\times10^{48}$ erg. The luminosity of the Sun is 3.8×10^{33} erg s^{-1}. Thus the gravitational energy release is enough to power the Sun for 10^{15} seconds, or about 3×10^7 years.

In the late 1800s, before nuclear processes were understood, the most efficient way known to produce energy was gravitational. The $\sim 3\times10^7$ year timescale is now known as the Kelvin-Helmholtz timescale, and because it was the *longest* that the Sun could last, it posed a problem in the late 1800s because by then it was clear that geological processes required much longer than that to operate. There is a story that when Ernest Rutherford announced the understanding that nuclear processes might provide a much more efficient energy source than gravitation and thus that the age issue might be solved, Lord Kelvin himself was in the audience. Rutherford was nervous, because Kelvin had imperiously denigrated the geologists, and Kelvin was the grand old man of British physics at that time. Rutherford apparently solved that problem by saying that "as Lord Kelvin wisely said, the age of the Sun is thirty million years *if* no more efficient energy source than gravitation could be found" [the actual quote was not recorded, but it was something like that]. So that's how nuclear fusion saved the Sun.

But the Kelvin-Helmholtz time *does* have a use in stellar astrophysics. Think about the protostellar phase, before the center of the star is hot and dense enough to start fusion. The gas that forms the star slowly contracts and releases energy and, as it turns out, the phase basically lasts the Kelvin-Helmholtz time. We can even go further with that insight: if we assume that the luminosity is roughly constant throughout the protostellar phase, then we would expect that the gas cloud will contract rapidly when the cloud is large and spend most of its time at small radii. That's because the gravitational energy released is $E_{\rm grav} \sim 1/R$ for radius R, and since the time is $T \sim E/L$ for luminosity L, we would expect that $T \sim 1/R$. Indeed, more sophisticated models show that the initial contraction is fast. There's a limit, though: the cloud can't contract faster than its free-fall time, which from Kepler's third law is $T_{\rm ff} \sim \sqrt{R^3/(GM)}$. For large enough clouds we expect the free-fall time to be a limiting factor, and it is.

But there's more! Let's now think about gas giants such as Jupiter. Jupiter's luminosity is more than double what it gets from the Sun, and the energy being released is gravitational. A recent estimate of Jupiter's internal luminosity is $L_J = 4.6 \times 10^{24}$ erg's^{-1}, and the order-unity estimate of its gravitational energy is $GM_J^2/R_J = 3.7 \times 10^{43}$ erg, so the timescale is $3.7 \times 10^{43}/4.6 \times 10^{24} = 8.0 \times 10^{18}$ s $\approx 2.6 \times 10^{11}$ yr. This is much longer than the 4.6×10^9 yr age of the Solar System. Thus gravitational energy release can easily last to the present time.

Indeed, the distinction between gas giant planets, brown dwarfs, and stars is that (1) stars are powered during most of their lives by fusion of ordinary hydrogen (just one proton in the nucleus) into helium, (2) brown dwarfs are powered during part of their lives by fusion of deuterium (heavy hydrogen, with one proton and one neutron in the nucleus) into helium (this can proceed at lower temperatures than fusion of ordinary hydrogen), and (3) gas giant planets never have interior temperatures high enough for any fusion, and therefore are powered mainly by gravitational energy release.

What about terrestrial planets such as the Earth? Early in their lifetimes, gravitational energy release is important. In fact, the collisions of planetesimals, if you think about it, release gravitational energy! However, after that phase is over, at the temperature and composition of terrestrial planets, they can't contract very easily and thus they have very little ongoing generation of energy through gravity. Instead, they just cool off (and their surface temperatures are determined by the illumination they get from the Sun). These are the sorts of processes you need to consider when you think about energy sources.

But there's more! When a massive star (more than 8 $M_\odot$ at birth; we'll learn more later) evolves, the core fuses to more and more massive elements with shorter and shorter times involved, until it gets to iron and can no longer extract energy from fusion (again, we'll learn more later). The core builds up until it collapses, and that collapse produces a neutron star with a mass $M_{\rm NS} \sim 1.5\ M_\odot$ and radius $R_{\rm NS} \sim 10$ km (yep, we'll learn more about all this later, too). The net result is that although the star emits $\sim$ few $\times\ 10^{51}$ erg by fusion in its lifetime, the final collapse emits $\sim GM_{\rm NS}^2/R_{\rm NS} \sim$ few $\times\, 10^{53}$ erg, i.e., about 100 times as much energy as was emitted during the entire lifetime of the star! This collapse, in fact, releases so much energy that it causes the star to explode in a supernova. Thus gravitational energy release is critical to the evolution of a high-mass star.

One final comment is that $E \sim GM^2/R$ is relevant for other sources of energy as well. For example, consider rotation. A star can rotate arbitrarily slowly, so the minimum available rotational energy is zero. But what about the maximum? The fastest that a star (which we assume is gravitationally bound) can rotate is the angular velocity at which the matter at the stellar surface is in orbit. From Kepler's third law, that angular velocity is $\Omega = \sqrt{GM/R^3}$, which means that the rotational speed is $v = R\Omega = \sqrt{GM/R}$ and the rotational energy is $(1/2)Mv^2 \sim GM^2/R$. There are astronomical objects that are powered by rotational energy (e.g., pulsars), but the total energy release can't be greater than the gravitational energy release. What about temperature, i.e., energy that was inside the star and emerges as it cools off? The virial theorem tells us that that in equilibrium the kinetic energy is 1/2 times the gravitational binding energy, so that is also of the order of GM^2/R. It also turns out, for less obvious reasons, that the maximum magnetic energy is also less than GM^2/R. All in all, for multiple energy sources, GM^2/R is a good formula to keep in mind.

In this lecture we have considered several non-fusion energy sources, but for completeness here are two others that have astronomical applications.

Crystallization

When a liquid cools enough to become a solid, the solidification process releases energy, which we will call crystallization even though the solid does not actually have to become a crystal. Like with chemical processes such as burning (or digestion!), one can again see that things are more or less electronic, in that it's electrostatic interactions that determine the energy. In more detail, it is typical that at some melting temperature (and thus crystallization temperature) T_m, the amount of energy emitted per ion is $\sim kT_m$. A typical freezing temperature for metals is a few thousand degrees, so 0.1–1 eV per ion is released, which is even less than for burning. Nonetheless, there are some astronomical objects for which crystallization is important. One is very old and cool white dwarfs, where the interior is in the process of solidifying. Another is a terrestrial planet such as the Earth. But these have to be pretty low-energy for crystallization to be important.

Fission

Fission releases approximately 1 MeV per nucleon, or about 10^{-3} of the rest mass energy. For the Sun the rest mass energy is about 2×10^{33} g c^2, or 2×10^{54} erg, so 0.1% of that is 2×10^{51} erg, which is enough. So why couldn't fission be a viable mechanism? Two reasons. First, it would require that a large fraction of the Sun, at least 25% by mass, would be made up of heavy elements such as uranium(!). But in reality uranium and other heavy elements make up a tiny fraction of elements. Even more conclusively, fission is a process that has a critical mass, and you'd better believe that a solar mass exceeds that critical mass! Therefore, if you did set up that much uranium, it would blow itself to bits within a ridiculously short time. So no dice on this mechanism for the Sun. However, there *are* some astronomical bodies for which fission is an important energy source: the Earth, for one.