Equations of state: degeneracy and interactions

Fermi-Dirac particles: Suppose we have a particle such as an electron, proton, or neutron which has a spin of 1/2 and is therefore a fermion. Let the energy reference level be mc^2 (again, other choices are possible!). The degeneracy is 2 (=2× spin plus 1), so the number density is

$$n = \frac{8\pi}{h^3}\int_0^\infty \frac{p^2\,dp}{e^{[-\mu+mc^2+E(p)]/kT}+1}\ . \tag{1}$$

In general, after we subtract the mc^2 reference energy $E(p) = \sqrt{m^2c^4+p^2c^2} - mc^2 = mc^2\left[\sqrt{1+(p/mc)^2}-1\right]$ and the velocity is

$$v(p) = \frac{\partial E}{\partial p} = \frac{p}{m}\left[1+\left(\frac{p}{mc}\right)^2\right]^{-1/2}\ . \tag{2}$$

As always, we should do some simple tests to determine whether this equation is correct, or at least not obviously incorrect. In the nonrelativistic limit $p\to 0$, we find that $v = p/m$, i.e., $p = mv$, which is correct. In the ultrarelativistic limit $p\to\infty$ we find that $v = (p/m)/(p/mc) = c$, which is correct. From this formula, and from what would be generally logical, the rough boundary in momentum between relativistic and non-relativistic is when $p\approx mc$.

Completely degenerate gas: Informally, degeneracy is when the density is high enough that particles start encroaching on each other's states. This has different implications for fermions than it does for bosons. For fermions this leads to Pauli exclusion and thus the fermions are forced into higher energy states, whereas with bosons this leads to multiple occupation of the same state. The result for bosons is a host of phenomena, such as lasers, Bose condensation, superfluidity, and superconductivity. Some interesting properties can be derived from the fact that the particles participating in these phenomena are in a single state. For example, superfluids can't rotate because the particles would be moving with different velocities and hence would not be in the same state. Instead, any rotation in a superfluid is quantized in vortices of normal fluid.

Complete degeneracy occurs when $kT\to 0$, or more precisely when kT is much less than $-\mu+mc^2+E(p)$ (given that we are using mc^2 as our reference energy level). For fermions the interesting part of the integrand is

$$F(E) = \frac{1}{e^{[E-(\mu-mc^2)]/kT]}+1}\ . \tag{3}$$

Ask class: what are the values for $kT\to 0$ when the exponent is positive and when it is negative? We see that $F(E) = 0$ or 1 for $kT\to 0$, depending on whether, respectively,

$E > (\mu - mc^2)$ or $E < (\mu - mc^2)$. In this case, therefore, there is complete occupation up to the "Fermi energy" $E_F = \mu - mc^2$ and no occupation beyond that. When kT is finite, the transition between occupation and no occupation occupies a width $\sim kT$ in energy. Only these particles can interact, and this fact has great importance for processes such as energy transfer. Specifically, it means that only a fraction $\sim (kT/E_F)$ of particles can interact, so mean free paths are a lot longer than one would imagine. **Note:** presaging our discussion of opacities, energy transfer tends to be dominated by whatever process can transport energy over a long distance rapidly. If there is a lot of scattering/absorption, this slows down the carrier particles. In degenerate matter, electrons travel a long way and hence conduction can be important; similarly, in a metal the periodicity of the potential cuts down on interactions and allows the electrons to travel long distances before they interact.

Remember that to squiggle order we can obtain the Fermi momentum from the uncertainty principle: $\Delta p \Delta x > \hbar$, so for a number density n we have $p_F \sim \hbar n^{1/3}$. A dimensionless parameter $x = x_F \equiv p_F/mc$ is often used to characterize how relativistic the Fermi energy is. Therefore,

$$E_F = mc^2 \left[(1 + x_F^2)^{1/2} - 1 \right] \ . \tag{4}$$

Hence the chemical potential is $\mu_F = E_F + mc^2$, which is the total energy of the most energetic particles in the system. That is, it's the "energetic cost" of adding another particle of this type.

Ask class: given this $kT \to 0$ approximation and the consequences for occupation number, how does the integral for the number density simplify? In that case, we simply integrate up to p_F, with an integrand of $F(E) = 1$, and then stop because there is no more occupation. Therefore,

$$n = \frac{8\pi}{h^3} \int_0^{p_F} p^2 \, dp = \frac{8\pi}{3} \left(\frac{h}{mc} \right)^{-3} x_F^3 \ . \tag{5}$$

Numerically, and dropping the "F" subscript,

$$n = 5.9 \times 10^{29} \left(\frac{m}{m_e} \right)^3 x^3 \, \mathrm{cm}^{-3} \ . \tag{6}$$

In density units, you have $\rho/\mu_e \approx 10^6 x^3$ g cm^{-3} for electrons (usually $\mu_e \approx 2$ in WD). The line between nonrelativistic and relativistic is approximately $x = 1$ (as can be seen from the $x = p/mc$ definition), which is $\sim 10^6$ g cm^{-3} for electrons. This is a useful number to remember: when the density exceeds 10^6, the electrons are relativistic.

If the electrons are nonrelativistic, then $E_F = p_F^2/2m$ and, since $p_F \sim \rho^{1/3}$ it therefore means that $E_F \sim \rho^{2/3}$. In the ultrarelativistic limit $E_F = p_F c \sim \rho^{1/3}$. Given this, we'd like an idea of when the Fermi energy is "important". That means that we should compare it with other relevant energies, such as the thermal energy and the Coulomb energy. It is useful to keep in mind that because $m_e c^2$ is about 511 keV, this corresponds to about 6×10^9 K.

For neutrons the transition density to a relativistic Fermi energy is 6×10^{15} g cm^{-3}, which substantially exceeds nuclear density of 2.7×10^{14} and is several times larger than is expected to exist even in the cores of neutron stars, so neutrons are mostly nonrelativistic.

In a similar way, the Fermi pressure and Fermi internal energy density for strongly degenerate particles can be computed by integrating up to the sharp cutoff p_F. We find:

$$P_e = \frac{\pi}{3}\left(\frac{h}{m_e c}\right)^{-3} m_e c^2 f(x) \equiv A f(x) \ , \tag{7}$$

where $f(x) = x(2x^2-3)(1+x^2)^{1/2} + 3\sinh^{-1} x$, and

$$E_e = A g(x) \ , \tag{8}$$

where $g(x) = 8x^3\left[(1+x^2)^{1/2} - 1\right] - f(x)$. Limiting forms are:

$$\begin{aligned} f(x) &\longrightarrow \tfrac{8}{5}x^5 - \tfrac{4}{7}x^7 + \cdots \ , \qquad x \ll 1 \\ &\longrightarrow 2x^4 - 2x^2 + \cdots \ , \qquad x \gg 1 \end{aligned} \tag{9}$$

and

$$\begin{aligned} g(x) &\longrightarrow \tfrac{12}{5}x^5 - \tfrac{3}{7}x^7 + \cdots \ , \qquad x \ll 1 \\ &\longrightarrow 6x^4 - 8x^3 + \cdots \ , \qquad x \gg 1 \end{aligned} \tag{10}$$

So far we've considered completely degenerate material, and we'll go back to that in a minute, but should also mention that a rough boundary between nondegenerate and degenerate matter is when $E_F > kT$. This allows us to answer the burning question...

For Perspective: am I degenerate? In the old days we'd figure this out by considering my deeds and bad habits, but now we can answer it mathematically! To get our answer we need to figure out the Fermi energy of my constituents, and then compare it to my thermal energy. **Ask class:** if there are plenty of free particles of all kinds, what kind of particle would be degenerate first? Electrons, because they have lower mass and $E_F \propto 1/m$ for nonrelativistic degeneracy. **Ask class:** is the nonrelativistic limit the correct one? Yes, because 10^6 g cm^{-3} is the rough boundary for relativistic degeneracy, and I'm nowhere near that!

In the examples above we've discussed matter that is completely ionized, so that electrons are free to move around as they will. However, in me the electrons are mostly not free. Instead, typically there are ions. So, let's calculate first what the Fermi energy is assuming the dominant species is a molecule of some sort. What is the most common molecule in me? Water, of course. Water has an atomic weight 18 times that of hydrogen, which we will round to about 20 times that of the neutron. The critical density at which the Fermi energy becomes relativistic goes like M^3, so for water it is about $20^3 \approx 10^4$ times that for neutrons, or about 6×10^{19} g cm^{-3}. Below this density the Fermi energy is nonrelativistic, and therefore goes like $p^2 \sim n^{2/3}$. At my density of ~ 1 g cm^{-3}, the Fermi energy is therefore $\sim 10^{-13}$ times the rest mass energy of water, or $10^{-13}\times 20$ GeV$=2 \times 10^{-3}$eV. The equivalent temperature for 1 eV is about 10^4 K, so this equates to about 20 K versus about 300 K for my temperature. Sadly, most of my mass is not degenerate! Of course, this is also true for, say, a white dwarf, where the mass is dominated by nondegenerate nucleons but the degenerate electrons provide the pressure.

But there may still be hope for me! Suppose that I have some small fraction of free electrons running around in me. In particular, suppose that there are about 10 electrons per molecule, and that about 1% of molecules have donated 1 electron to the general environment. The density of free electrons is therefore 10^{-3} times the density it would be if all atoms were completely ionized. For the purpose of this calculation, therefore, it's as if I were completely ionized but had a density of about 10^{-3} g cm^{-3}. Using the same approach as before, we know that for electrons the density at which relativistic degeneracy starts is about 10^6 g cm^{-3}, and that below this the Fermi energy scales as $p^2 \sim n^{2/3}$. Therefore, at 10^{-9} of this density the energy is 10^{-6} of the electron rest mass energy, or 0.5 eV. This equates to $\sim$5000 K, meaning that my electrons would be degenerate by a factor of more than 10! Woohoo! Unfortunately, J. Norman Hansen, professor of chemistry and biochemistry at the University of Maryland, told me that in biological systems free electrons essentially don't exist, because as soon as one would be stripped off of a molecule it would go to another one, and hence electrons spend time in one orbital or another. This is also the conclusion we'd reach from the Saha equation: the ionization equation, modulo factors of little interest, would be something like

$$\frac{y^2}{1-y} \propto e^{-3\times 10^4/T} \tag{11}$$

for typical electron ionization energies of $\sim$2 eV, meaning that at 300 K the exponential is like e^{-100}, so there is virtually no ionization.

Thus, tragically, I'm not degenerate :). I'm crushed, but let's move on to white dwarfs.

White Dwarfs: Sirius B is the stellar companion of Sirius A, which is the brightest star in our night sky. It has the distinction of providing an amazing confirmation of Newton's laws (in the 1800s it became clear that Sirius A wobbles in its motion, so the 1862 discovery of Sirius B by Alvan Clark confirmed Newton's law of gravity outside our Solar System). But unknown to the scientists of the time, Sirius B was also the first-discovered white dwarf, which is held up against gravity not by radiation and gas pressure gradients but by gradients of degeneracy pressure. By the 1920s it became clear that white dwarfs could not be explained classically; a quantum mechanical explanation was finally provided in 1926 by R. H. Fowler.

In 1930, Subramanyan Chandrasekhar (known informally as Chandra) worked on the fundamentals of the structure of white dwarfs, coming up with two major results while on a 19-day boat trip from Madras, India, to England to work at Cambridge. One was the mass-radius relation $R \sim M^{-1/3}$. The other was completely unexpected: white dwarfs have a maximum mass. We'll obtain that result using a simpler argument, due to Landau, than was originally provided by Chandrasekhar.

We start with a reminder that when the Fermi momentum and energy are in the non-relativistic regime, then $E_F \approx p_F^2/(2m)$ for a particle of rest mass m, and since $p_F \sim \hbar n^{1/3}$, for a star with N degenerate particles in a radius R (so that $n \sim N/(4\pi/3R^3) \sim N/R^3$) we have $p_F \sim \hbar N^{1/3}/R$. Thus in the nonrelativistic case the Fermi energy per particle is $E_F \sim \hbar^2 N^{2/3}/(2mR^2)$.

In addition to the Fermi energy we have the gravitational potential energy, which is negative. Note that although the Fermi energy we are considering is that of the electrons, the gravitational potential energy will be dominated by the heavier baryons (i.e., the neutrons and protons). Say that the baryonic mass is m_B and pretend that there is one baryon per electron (in reality, for nuclei heavier than hydrogen, it's more like two baryons per electron; we'll get to that later). Then the gravitational potential energy per electron is $E_G \sim -GMm_B/R \sim -GNm_B^2/R$ because $M \sim Nm_B$ under our current approximations. This means that the total energy per electron in the nonrelativistic case is

$$E_{\rm tot} = E_F + E_G \sim \frac{\hbar^2 N^{2/3}}{2m_d R^2} - \frac{GNm_B^2}{R} \ , \tag{12}$$

where now we write m_d for the mass of the degenerate particle, i.e., $m_d = m_e$ for degenerate electrons. We can find the equilibrium radius $R_{\rm eq}$ by taking the derivative of $E_{\rm tot}$ with respect to R and setting it equal to zero. This gives $R \sim N^{-1/3}$, and since $M \sim N$ we find $R \sim M^{-1/3}$. As we add mass to a white dwarf, the equilibrium radius decreases like $R_{\rm eq} \sim M^{-1/3}$ as long as electron degeneracy is nonrelativistic.

But given that $\rho \sim M/R^3$, the equilibrium average density goes like $M/\left(M^{-1/3}\right)^3 \sim M^2$. Thus higher-mass white dwarfs have higher densities. Higher densities means higher Fermi momenta. Eventually, the typical electron Fermi momentum in a white dwarf will become relativistic. What happens then?

For relativistic Fermi momenta, $E_F \approx pc \sim \hbar n^{1/3} c \sim \hbar c N^{1/3}/R$. Under the same rough assumptions as before, the total energy per degenerate particle is now

$$E = E_F + E_G = \frac{\hbar c N^{1/3}}{R} - \frac{GNm_B^2}{R} = \frac{1}{R}\left(\hbar c N^{1/3} - GNm_B^2\right) \ . \tag{13}$$

Unlike in the nonrelativistic case, here both terms scale as R. Again, suppose that we want to find the equilibrium radius $R_{\rm eq}$, which we do by minimizing the total energy as a function of R. Suppose first that N is small. Then E is positive, so that E can be decreased by increasing

R, and at some point the fermions become nonrelativistic so that $E_F \sim p^2 \sim 1/R^2$, and a stable equilibrium is possible. However, if E is negative then E can be decreased indefinitely by decreasing R, leading to instability and collapse. The boundary point in N is the boundary for stability, and comes at

$$N_{\rm max} \sim \left(\frac{\hbar c}{G m_B^2}\right)^{3/2} \sim 2 \times 10^{57} \; . \tag{14}$$

This gives a mass of about $1.5\,M_\odot$ for the maximum, which is indeed approximately the maximum possible mass for a white dwarf. Note, though, that this derivation is a bit fudged. We sloppily assumed that the number μ_e of baryons per degenerate fermion is 1, and we dropped a few factors along the way. If we do it more carefully, we find a maximum mass of $1.46(2/\mu_e)^2\ M_\odot$, which is about $1.35\ M_\odot$ for a white dwarf made of iron-56. Because the same chain of reasoning applies to neutron stars, for which $\mu_e \approx 1$ because the degenerate particles (neutrons) are the same as the heavy baryons, we'd expect that neutron stars would have maximum masses of about $6\ M_\odot$. Instead, the maximum mass, although not well known, is more like $2-3\ M_\odot$. Can you think of what effects might reduce the maximum mass, and make it much more uncertain for neutron stars than for white dwarfs?

Consequences for black holes: from Chandra's work it was known that a white dwarf more massive than about $1.4\,M_\odot$ could not exist, so initially this seemed to doom any star with an initial mass greater than this. However, it turns out that winds and the planetary nebula phase (to be treated later) remove a lot of mass, and up to $\sim 8\,M_\odot$ or so initial mass stars end up as white dwarfs. With a higher initial mass, however, the star can't end its life as a white dwarf. Instead, it collapses further, to a neutron star or a black hole, and the collapse releases enough energy that the star explodes in a core-collapse supernova.