Radiative opacities

Radiative opacity sources: What are the ways in which a photon can interact? Interaction can happen with free electrons, atoms, molecules, or dust. Radiation can also interact with protons/nuclei, magnetic fields, other photons, etc., but in this class we'll ignore those for the most part. For free electrons, we have electron scattering (not an absorption) and free-free absorption (which is also called inverse bremsstrahlung). Scattering means that the photon retains its identity (in a somewhat ill-determined way), whereas absorption means that the photon is destroyed and later another one is emitted.

For atoms, we have transitions between atomic levels (bound-bound absorption) and ionizations (bound-free absorption), which kick an electron initially bound to an atom to a free state. When the ionization fraction is low (i.e., when almost all electrons are bound to atoms) then bound-bound and bound-free opacities take center stage, whereas when the ionization fraction is high, opacities involving free electrons are the most important. Molecules and dust are more important for lower-energy photons. The same is true for H^- opacity (i.e., transitions involving a proton with two electrons orbiting around it), which is important for some stars.

Scattering from free electrons: Let's start with Thomson scattering, in which a photon scatters off of an electron and, in the initial electron rest frame, the photon does not change its energy after scattering (equivalently, the electron experiences no recoil). This is a good approximation for photon energies much less than the mass-energy of an electron. In that limit the total cross section is independent of the photon energy:

$$\sigma_T = 6.65 \times 10^{-25}\,\mathrm{cm}^2\ . \tag{1}$$

Thomson scattering is not isotropic: the differential cross section for a direction θ away from the initial propagation direction of the photon is $d\sigma/d\Omega \sim (1+\cos^2\theta)$ (recall that Ω in these contexts refers to the solid angle). Thus, photons undergoing Thomson scattering are not exactly doing a random walk, since they have a greater chance of going forward or backward than to the side. This, however, makes little difference to the propagation of a photon.

We can think of Thomson scattering as the oscillating electric field of the photon causing the electron to wiggle back and forth and hence radiating. Given this picture, should we expect it to be easier or harder to do the same thing with a proton? Harder, because the proton has more inertia, so under the influence of a given electric field a proton accelerates less than an electron and therefore the proton radiates less. With that in mind, should the Thomson scattering cross section be more or less for protons than for electrons? It should be less, and is: the Thomson cross section is proportional to the mass of the particle to the -2 power, so scattering from ions is almost always negligible.

As we indicated, pure Thomson scattering is elastic, so photons retain their energy exactly. But is this true in reality? In particular, think of the process in the frame in which the electron was initially at rest. There must be *some* recoil, so the photon after scattering has to have lost some energy. When we work it out we find that after scattering the photon has on average increased its wavelength by of order the wavelength of a photon whose energy is the electron rest-mass energy (i.e., the electron Compton wavelength):

$$\lambda_f - \lambda_i = \lambda_c(1 - \cos\theta) = \frac{h}{mc}(1 - \cos\theta) \; . \tag{2}$$

Here λ_i is the initial, and λ_f the final, wavelength of the photon in the original electron rest frame. In this expression, θ is the angle between the final direction and the initial direction; note that when $\theta = 0$ the photon is undeflected and thus, as we would expect, does not change its wavelength.

The previous discussion focused on scattering as seen in the initial rest frame of the electron. In that frame, the photon always loses energy after scattering. However, in the "lab frame" (i.e., a frame in which the electron might initially have been moving) the electron, if moving fast enough, can increase the energy of the photon after scattering; this is called inverse Compton scattering.

To see this most easily, suppose that in the lab frame the electron initially moves with a speed v, and the photon initially has an energy E and is moving head-on to the electron. In the electron rest frame, the photon energy is blueshifted, so in the electron rest frame the photon energy is $E\gamma(1 + v/c)$, where $\gamma = (1 - v^2/c^2)^{-1/2}$ is the Lorentz factor. Say that the photon bounces back 180° after scattering with the electron, and that in the electron rest frame the photon's energy does not change (because it is much less than the electron rest-mass energy $m_e c^2$, where m_e is the mass of the electron). Then, after scattering, in the electron rest frame the photon energy is still $E\gamma(1 + v/c)$. But Doppler shifting back to the lab frame, the photon's energy is now $E\gamma^2(1 + v/c)^2 = E(1 + v/c)/(1 - v/c)$, so the photon has gained energy in the lab frame. Now, we should also understand that the photon does not *always* gain energy in the lab frame. For example, if we were to redo the analysis above but assume that the photon hits the electron tail-on, we would find that the final energy in the lab frame is $E(1 - v/c)/(1 + v/c)$, so in that case the photon would lose energy as seen in the lab frame. When both the electrons and photons are in isotropic thermal distributions, equilibrium is reached (not surprisingly) when the radiation and electron temperatures are equal.

The recoil effects also modify the scattering cross section somewhat and make it frequency-dependent. For initial photon energy ϵ_i, final photon energy ϵ_f, and a photon direction after scattering that is θ away from the initial direction, the differential cross section in the initial electron rest frame is

$$\frac{d\sigma}{d\Omega} = \frac{r_0^2}{2}\frac{\epsilon_f^2}{\epsilon_i^2}\left(\frac{\epsilon_i}{\epsilon_f} + \frac{\epsilon_f}{\epsilon_i} - \sin^2\theta\right) \ , \tag{3}$$

where $r_0 = e^2/m_e c^2 \approx 2.8 \times 10^{-13}$ cm is the classical radius of the electron (but remember, quantum mechanics rules; you should not think of electrons as hard spheres with this radius!). This is the Klein-Nishina formula, and it causes significant deviations from the Thomson case only when the photon energy as measured in the electron rest frame is comparable to or larger than the electron rest mass energy. If we define $x \equiv \hbar\omega/m_e c^2$, then the approximate total cross section is $\sigma \approx \sigma_T(1 - 2x)$ when $x \ll 1$. Given that $m_e c^2 = 511$ keV, the deviation from the Thomson cross section is therefore only a big deal for $\hbar\omega$ more than about 100 keV. Scattering is often a minor player for photon energies below $\sim$keV or so, corresponding to $\sim 10^7$ K, because at lower energies absorption processes dominate.

One last bit of scattering to mention for completeness is cyclotron scattering. This occurs when the electron cyclotron energy $\hbar\omega_c = \hbar eB/m_e c \approx 10(B/10^{12}\,\mathrm{G})$ keV is in the frequency range of interest. At such frequencies, the incoming photon can resonate with the cyclotron frequency. The cross section for cyclotron scattering, at resonance, is $\sim 10^6$ times the Thomson cross section, so in the right circumstances (the atmosphere of a neutron star or of a very magnetic white dwarf) it can have a major effect on the emitted spectrum, but for normal stars the cyclotron energy is way below the energies of interest and thus cyclotron scattering has a negligible effect on the spectrum.

Bound-bound absorption This is when an electron bound in an atom absorbs a photon and consequently makes a transition to a higher bound level. An electron in such a higher level has a finite lifetime to decay Δt. From a version of the uncertainty principle, $\Delta E \Delta t \geq \hbar/2$, this means that the energy width Γ of the transition is $\Gamma \sim \hbar/\Delta t$. Applying time-dependent perturbation theory, the cross section of a transition from initial state s to final state k as a function of energy can be written in the form

$$\sigma(\omega) = \frac{2\pi^2 e^2}{m_e c} f_{\mathrm{ks}} \frac{\Gamma/2\pi\hbar}{(\omega - \omega_{\mathrm{ks}})^2 + (\Gamma/2\hbar)^2} \ . \tag{4}$$

Here f_{ks} is the "oscillator strength", and this functional form is called a Lorentzian. There is a sum rule saying that the sum of the oscillator strengths of all transitions of an electron from a given energy level is 1; we do have to be cautious, though, because emission counts negatively for the sum. Most oscillator strengths are less than or of order 1.

As an example, take the Lyα resonance of hydrogen, where an electron in the 1s state is absorbed into the 2p state. This has an oscillator strength of 0.42 and a peak cross section of 7.2×10^{-11} cm. This is *enormous*, a good 10^{14} times the Thomson cross section. However, it is important over only a very small range in energy. The lifetime is about 10^{-9} s, so the energy width is $\hbar/10^{-9}\ \mathrm{s} \approx 10^{-6}$ eV.

Generally, therefore, bound-bound transitions have extremely high cross sections over extremely small regions of the spectrum. Given that the Rosseland mean is harmonically weighted, I would have expected that bound-bound transitions, although producing all of the lines that we see in spectra, would be unimportant in energy transport. People in the 1960s and earlier thought so, too, but it turns out that the total opacity in nonionized portions of the Sun can be increased by a factor of 2 or 3 by bound-bound transitions. This is because with all the transitions available from metals, the wings of the lines overlap.

Bound-free absorption

The process here is that a photon ejects an electron from a bound state inside an atom. There is thus a threshold energy for the process; the photon needs to have at least as much energy as the ionization energy. The cross section peaks at about the ionization energy, and then decreases at higher energies, approximately like ν^{-3} for a photon frequency ν above the threshold. Above the threshold, the cross section for hydrogenic atoms (meaning ones with only one electron, e.g., singly-ionized helium, five-times-ionized carbon, and so on) is

$$\sigma_{\rm b-f} \approx 2.8\times 10^{29}\frac{Z^4}{n^5\nu^3}g\ {\rm cm}^2\ , \tag{5}$$

where $g=g(\nu,n,l,Z)$ is the Gaunt factor (which basically is a correction factor between the correct quantum mechanical answer and the simplified semiclassical approximation), which is typically close to unity, n is the principal quantum number (1 for the ground state, 2 for the next excited state, and so on), Z is the electric charge of the nucleus in units of the proton charge, and ν is measured in Hertz. At the threshold of $n=1$ ionization this is close to 10^{-17} cm^2 for hydrogen (which has $Z=1$), which is much less than the peak of a bound-bound transition but much more than the Thomson cross section. Using the ν^{-3} shape, bound-free absorption becomes competitive with Thomson scattering for half-ionized hydrogen when the photon energy is about 3 keV (remember that we compare *opacities*; for fully ionized hydrogen there is no bound-free absorption, and for fully neutral hydrogen there is no Thomson scattering, so we compare at half ionization).

To get the opacity requires multiplying this by the number of ions in this state per gram. An expression often used (e.g., in our textbook) is

$$\kappa_{\rm b-f} \approx 4\times 10^{25} Z(1+X)\rho T^{-3.5}\,{\rm cm}^2\,{\rm g}^{-1}\ , \tag{6}$$

where Z is now the fractional abundance of metals. But to paraphrase Clint Eastwood, an equation's got to know its limitations! This expression would predict zero bound-free absorption in pure hydrogen, if used blindly, because then $Z=0$. The real point is that the elements with ionization energies comparable to the temperature will contribute most, and at most temperatures of interest in stars the hydrogen is fully ionized. Therefore, as the frequency of the photons and the temperature go up, one metal/atomic state takes over where another has left off. The net result in the interstellar medium is that at far UV to X-ray energies the opacity is dominated by bound-free absorption, and actually ends up having the approximate form $\kappa \sim \nu^{-8/3}$ (rather than the perhaps-expected ν^{-3} dependence) because of the relative abundances of elements. This means that at low energies (below around 0.5 keV or so, depending on the column depth, which is the number of atoms per square centimeter toward the source) the spectrum appears cut off to us.

This expression for the opacity, with a factor $\rho T^{-3.5}$, is called a Kramers opacity. It also applies to free-free absorption, as we will see, and is a very generally useful form of the opacity for much of the interior of stars.

Free-free absorption

In this process a free electron passes by an ion of charge Z, scatters off of it, and absorbs a photon. This is the inverse process to bremsstrahlung, or "braking radiation", in which an electron accelerated by an ion radiates a photon. Conservation of energy in bremsstrahlung means that

$$\frac{p_k^2}{2m_e} = \frac{p_s^2}{2m_e} + \hbar\omega \ , \tag{7}$$

where p_k is the initial momentum of the electron and p_s is the final momentum. Here we are assuming nonrelativistic electrons, as we can tell from the $p^2/2m$ form of the energy. The presence of an ion is very important. For example, an electron by itself, in free space, cannot emit a photon because it is impossible to conserve both energy and momentum in that situation.

The cross section for free-free absorption around an ion of charge Z is

$$\sigma_{\mathbf{f-f}} = 3.7 \times 10^8 \frac{Z^2 n_e g_{\mathrm{f-f}}}{T^{1/2}\nu^3}\,\mathrm{cm}^2 \ . \tag{8}$$

A reasonable approximation for the free-free Rosseland mean opacity is

$$\kappa_{\mathrm{f-f}} \approx 10^{23} \frac{\rho}{\mu_e} \frac{Z_c^2}{\mu_I} T^{-3.5}\,\mathrm{cm}^2\,\mathrm{g}^{-1} \ , \tag{9}$$

where μ_e and μ_I are the mean molecular weights that we've used before, *but* remember that only free electrons contribute!

The Total Radiative Opacity

At this point, let's step back and think about what this means. In principle, a photon propagating through gas will have some probability of interacting in all of the ways above (plus others we haven't discussed as much, such as pair production for high energy photons, and so on). We need a way to determine what dominates when.

To make the point clearer, suppose that we suppress all photon interaction mechanisms except for one. To be definite, say that for the moment we only consider scattering. Let's consider photons of a given energy. What factors determine the mean free path ℓ of those photons? The only factors can be cross section and the number density of scatterers. Specifically, $\ell = 1/n\sigma$, where n is the number density of only the gas particles that can scatter. This means, for example, that if we have a gas of 100% neutral hydrogen atoms, then the mean free path to Thomson scattering is infinite, because there are no free electrons from which the photons would scatter.

This comes back to the difference between *cross section* and *opacity*. The cross section says (in this case): suppose you have free electrons. What is the surface density (in cm^{-2}) of those free electrons so that only a fraction e^{-1} of photons go unscattered? That's just $\Sigma = 1/\sigma$. But this does *not* tell you how important scattering is! For that, you need to know the number density of the scatterers. Opacity is expressed in terms of $\kappa = \text{cm}^2\ \text{g}^{-1}$. This means that the number of scatterers (each with cross section σ) per gram of material is κ/σ.

Now suppose you have hydrogen gas that is 100% ionized. How important do you expect bound-free ionization to be? Totally unimportant, because there are no bound atoms! How about bound-bound? Same thing. Of the processes we've considered, only free-free absorption and scattering could possibly contribute.

Now let's suppose that we have photons of a given energy. We've computed the opacities for all the different processes. The processes with high opacities will dictate the total opacity, because those interactions will occur first. At a given frequency, high opacities win.

Now, suppose that we have a flat input spectrum of photon energies. We send it into a region of gas, where the total opacity varies as a function of energy. Should we expect most of the radiation transferred through this region to be at frequencies of high opacity or of low opacity? Low opacity, because it is easier to propagate photons that way. That's why in optically thick plasmas (such as the Sun!), lines tend to be dark, because there's less emission there. In fact, when one considers the "effective" opacity over a spectrum in an optically thick region, one usually takes a harmonic average (that is, weighted by the reciprocal of the opacity). As we encountered earlier, this is called a Rosseland mean.

Summary: opacity, not cross section, determines importance of processes. At a given photon energy, high opacities dominate. Over a spectrum, low opacities dominate.