**Other radiative opacities and conduction**

**$H^-$ Opacity**

In the last class we talked about atomic opacities. Perhaps surprisingly, for cool stars, molecular and dust opacities play major roles. In addition, the negative hydrogen ion, $H^-$, can be extremely important (for example, it is the most important source of opacity in the solar photosphere). The extra electron in the hydrogen atom is bound, but not by much: only 0.75 eV. This means that the highest temperature at which we would expect this to exist is given by $kT \sim 0.75$ eV, which is close to the photospheric temperature of the Sun. Note also that for this ion to be formed, neutral hydrogen has to capture a free electron, meaning that both neutral hydrogen and free electrons need to be present. Therefore, $H^-$ opacity is unimportant when there is no neutral hydrogen, or when the temperature is so low that everything is neutral. Metals, with more loosely bound outer electrons, can contribute to the free electron population and therefore are important for the $H^-$ opacity. At $T < 2500$ K or for low-metallicity stars, the $H^-$ opacity is small. For 3000 K$< T <$6000 K, $10^{-10} < \rho < 10^{-5}$ g cm$^{-3}$, and reasonable hydrogen and metal abundances, an approximate fitting formula for the opacity gives

$$\kappa_{H^-} \approx 2.5 \times 10^{-31} (Z/0.02) \rho^{1/2} T^9 \text{ cm}^2 \text{ g}^{-1} , \tag{1}$$

where as usual $T$ and $\rho$ are in cgs units. Tables of this opacity exist, and are to be preferred to the fitting formula for serious applications.

When the temperature is below about 3000 K, molecules and dust form and are extremely important for the atmospheric opacities of low-mass stars. For example, water (gaseous, of course!) has been found in sunspots, where the temperature is low(!). Maybe even more surprisingly, a prominent source of lines in the spectra of low-mass stars is titanium oxide (!!!), which despite the low abundance of titanium has the right combination of characteristics to make those lines. Molecular and dust opacities are extremely complicated because of their extra degrees of freedom (e.g., molecules, unlike atoms, can vibrate along molecular bonds and can rotate), and are still not especially well characterized for many transitions.

**Conduction**

We will now transition from energy transport by photons, to a type of energy transport by particles: conduction. In a general way, some of the same principles apply as they do for photons. For example, when there is a temperature gradient, particles from the hotter region can carry energy to the cooler region. As we discussed briefly earlier in the course, given that at a given temperature $T$ the radiation energy density scales as $T^4$, whereas the energy density of particles goes as $nkT$ for number density $n$, with all else equal we expect that at lower temperatures there will be a larger fraction of the total energy in particles than in radiation. That doesn't prove that conduction has to be important at low temperatures, but it suggests that we might look in that direction.

Now let's think about the differences between energy transport by radiation and by conduction. The two biggest fundamental differences are (1) electrons are fermions whereas photons are bosons, and (2) electrons have electric charge whereas photons do not.

First, a reminder about the fermion/boson difference. Fermions don't like other, identical, fermions. That's a bit anthropomorphic, but it reminds us that two identical fermions (such as two electrons, or two protons, or two neutrons) cannot occupy the same quantum state (position, momentum, spin up/down). In contrast, bosons (such as photons) like each other and are happy to occupy the same quantum state. The reason that this is relevant is that when you have degenerate electrons (which you recall means that the Fermi energy $E_F$ is greater than the thermal energy $kT$), only a small fraction $\sim kT/E_F$ of electrons can interact, because otherwise two or more electrons will occupy the same state.

This means that when you have strongly degenerate matter, from the standpoint of interactions it is as if the density is much smaller than it actually is, and thus the mean free path for electrons is longer than we might have guessed. If you scatter one electron with another in the process of conduction, then both the initial and final electron states have to be unoccupied. Thus degenerate matter has high thermal (and electrical) conductivity because the electrons can travel a long way. Nondegenerate matter does not have that advantage, although for matter with a lattice structure or high mobility of electrons (think about a crystal or a metal), electrons can also travel long distances. Thus in astrophysical settings, conductivity is most common when you have degeneracy or another circumstance where the electrons can travel great distances.

The electric charge of electrons (versus the neutrality of photons) comes in when we consider a region of high density and temperature next to a region of low density and temperature. The electrons in the high density/temperature region will move more quickly and with higher electron flux than the ones in the low density/temperature region. Therefore, there is a net flux of electrons. For photons that would be it, but for electrons an electric field would develop due to the charge imbalance, and would quite rapidly prevent any net flux of electrons. Therefore, the flux of charge in one direction must equal the flux of charge in the other direction. But then how can there be a net transfer of energy? The answer is that even though the net flux of particles is zero, the energies are not the same (one side is hot and the other is cold), so energy flux is nonzero.

Now let's think about which particles will contribute the most to conduction. In an ideal monatomic gas of temperature $T$, the energy per particle is $E = \frac{3}{2}kT$. Given that, and given that the kinetic energy is $(1/2)mv^2$, in thermal equilibrium lighter things (such as electrons) will travel faster than heavier things (such as ions). For example, if we consider pure hydrogen, because protons are $\sim 1800$ times more massive than electrons, the electrons move $\sim \sqrt{1800} \sim 43$ times faster than the protons. The ratio is even larger if the ions are heavier. This means that electrons dominate conduction, and that to a reasonable approximation you can think of a fast-moving electron gas and a nearly stationary background of ions.

For matter in most parts of normal stars, radiative energy transport is much more important than conductive energy transport. For example, if we have an ionized carbon gas at $T = 10^6$ K and $\rho = 1$ g cm$^{-3}$, then it turns out that the mean free path for an electron is $10^{-4}$ times the mean free path for a photon.

For degenerate matter, conductivity can be the main form of energy transport. We can get some insight into more details by asking: will electron scattering off of electrons be more or less common than electron scattering off of ions? If ions with nuclear charge $Z > 1$ have the same number density as the electrons, then ions are more important, because the cross section goes like $Z^2$. Thus even if the system is nondegenerate, you would think that the mean free path of the electrons would be determined by electron-ion scattering. If the electrons are degenerate then the ions dominate scattering even more, because in degenerate electrons both the initial and final state of the electron has to be unoccupied. As we said before, only a $kT/E_F$ of the electrons can interact in that situation. For electron-electron scattering, we need both the initial and final states for both electrons to be unoccupied. For electron-ion scattering, we need this only for one electron.

If the ions are in a perfect lattice, then the potential is exactly periodic and the electrons move as if they were free (this is why metals on Earth conduct so well; they are nearly periodic so the electrons move a long way). Therefore, electron-ion scattering depends on impurities and/or imperfections relative to a perfect lattice. This is one place where the lack of catalysis to equilibrium nuclear matter is important, because it increases resistivity dramatically! In any case, this is a reason why conductivity is so important in high-density things such as neutron stars: in much of the crust, the ions are in a lattice-like structure, so conductivity is huge.

To see what this means, suppose that you were in a hut in winter with terrible insulation. How would the temperature difference (inside to outside) compare with being in a hut with excellent insulation? The $\Delta T$ is much larger when the insulation is good. Similarly, if you had a situation in which the thermal conductivity was large, and you fixed the total energy flux, what would this mean about the temperature gradient compared to when the total conductivity was small? The temperature gradient is smaller when the conductivity is high (equivalently, when the opacity is low). This means that degenerate objects such as white dwarfs or neutron stars are close to isothermal in their interiors.

Now we will treat conductivity as in our textbook. We can use Fick's law of diffusion:

$$F_{\rm cond} = -D_e \frac{dT}{dr} \ . \tag{2}$$

You may note a difference from how we treated radiative diffusion: there, we used $d(aT^4)/dr$. The reason for the difference is that the energy content of the particles is like $kT$, whereas the energy density of radiation is like $aT^4$. It is convenient to recast this in a form that is similar to that for radiation, by defining a "conductive opacity"

$$\kappa_{\rm cond} = \frac{4acT^3}{3D_e\rho} \ . \tag{3}$$

The conductive flux from the diffusion equation is then

$$F_{\rm cond} = -\frac{4ac}{3\kappa_{\rm cond}\rho} T^3 \frac{dT}{dr} \ . \tag{4}$$

Using this, we can see another way how to combine the opacities: the energy fluxes should add, so $F_{\rm tot} = F_{\rm cond} + F_{\rm rad}$, meaning that $1/\kappa_{\rm tot} = 1/\kappa_{\rm rad} + 1/\kappa_{\rm cond}$. Therefore,

$$F_{\rm tot} = -\frac{4ac}{3\kappa_{\rm tot}\rho} T^3 \frac{dT}{dr} \ . \tag{5}$$

The diffusion coefficient has the general form

$$D_e \approx c_V v_e \lambda/3 \ , \tag{6}$$

where $c_V$ is the specific heat at constant volume of the electrons, $v_e$ is the typical speed of the electrons, and $\lambda$ is a typical mean free path. We can see why each factor is needed: the higher the specific heat, the greater the energy difference for a given $dT$; the higher the velocity, the faster the energy will be transported; and the longer the mean free path, the greater the temperature gradient that will be sampled.

Now let's go through a rough derivation of the dependences of those three factors for nonrelativistically degenerate electrons. *All of this assumes that the ion positions are uncorrelated with each other*; lattice effects can be substantial in a real star. The specific heat for degenerate electrons is proportional to $c_V \propto Tx_f(1+x_f^2)^{1/2}$, where $x_f \equiv p_F/(mc)$, where $p_F$ is the Fermi momentum. For nonrelativistic electrons $x_f \ll 1$, so to leading order this expression is something like $Tx_f$, and remember that $x_f \sim n_e^{1/3}$. **Note:** our textbook talks about $\rho/\mu_e$ instead of $n_e$, but I'm using $n_e$ for simplicity. So, that's the first factor: $c_V \propto Tx_f$.

How about the velocity? Remember that for degenerate electrons, only the electrons at the top of the Fermi sea can interact, meaning electrons within $kT$ or so of $E_F$. If the electrons are strongly degenerate then $E_F \gg kT$ and thus the energy of the interacting electrons is $\sim E_F$. That means, for nonrelativistic electrons, that $m_e v_e \approx p_F$. Because $x_f \propto p_F$, we find that $v_e \propto p_F \propto x_f \propto n_e^{1/3}$. This is the second factor.

Finally, what about the mean free path $\lambda$? Using our general formula, $\lambda = 1/\sigma_C n_I$, where $\sigma_C$ is the Coulomb scattering cross section and $n_I$ is the number density of the ions. For the cross section, as a rough approximation the usual estimate is a cross section corresponding to an impact parameter that will "significantly" change the path of the electron, meaning one in which the electrostatic energy is equal to the kinetic energy of the electron. Therefore, the impact parameter $s$ is given by

$$m_e v_e^2 \approx Z_c e^2/s \Rightarrow s \propto 1/v_e^2 \propto n^{-2/3} \tag{7}$$

for an ion charge $Z_c$. The cross section itself is $\sigma_C \propto s^2 \propto n^{-4/3}$. Therefore, $\lambda \propto n^{4/3}/n \propto n^{1/3}$, and the diffusion coefficient goes like $n$. Putting in all the constants and mean molecular weights, a decent approximation to the conductive opacity is

$$\kappa_{\rm cond} \approx 4 \times 10^{-8} \frac{\mu_e^2}{\mu_I} Z_c^2 \left(\frac{T}{\rho}\right)^2 \ {\rm cm^2\ g^{-1}}\ . \tag{8}$$

Just for fun, let's apply the same logic when the degeneracy is relativistic. In that case, the specific heat is $Tx_f^2 \propto Tn_e^{2/3}$. Since the speed is relativistic, $v \approx c$. For the characteristic impact parameter, we want to know when $Z_c e^2/s = E_F \approx p_F c \propto x_f \propto n_e^{1/3}$, so the impact parameter $s$ goes like $n_e^{-1/3}$, the cross section goes like $n_e^{-2/3}$, and the mean free path goes like $1/\sigma n \sim n^{-1/3}$. The diffusion coefficient then goes like $n^{1/3}$ instead of $n$, and the conductive opacity goes like $\kappa \sim T^2/\rho^{4/3}$. This expression has little use in practice because at densities great enough that degeneracy is relativistic, correlations between ions are usually very significant, causing substantial deviations from this dependence.

**Comparison of radiative and conductive opacities**

Recalling that to a *very* rough approximation the bound-free opacity is

$$\kappa_{\rm b-f} \approx 4 \times 10^{25} Z(1+X)\rho T^{-3.5} \tag{9}$$

(with everything in cgs as usual) how does this compare to the conductive opacity? Let's compare typical situations in the center of the Sun and in the center of a white dwarf. Suppose that in the center of the Sun we have $T \approx 10^7$ K and $\rho \approx 100$ g cm$^{-3}$. Then $\kappa_{\rm b-f} \approx 100$ cm$^2$ g$^{-1}$. In the center of a white dwarf you might have $T = 10^6$ K and $\rho = 10^6$ g cm$^{-3}$. Then $\kappa_{\rm cond} \approx 10^{-7}$ cm$^2$ g$^{-1}$, according to this formula (it will probably be even less). The net result is that the conductive opacity in very degenerate material is tiny, so to a decent approximation the whole interior can be considered isothermal.

**Tabulated opacities**

The simple expressions we've used for the different types of opacities are used to illuminate the physics, and in this respect are quite useful. However, when you get down to real modeling you never use them. Instead, you use the painstakingly compiled opacity tables from places like LANL or LLNL. Much of this relies in turn on atomic transition data, which is itself far from complete; by some estimates, half the lines in the solar spectrum are not identified! Still, for most purposes (other than for very cool stars) the tables are adequate.