Black Holes

We're now ready to start tackling the properties of black holes. As before, we'll start with a qualitative summary of their properties, then go into more quantitative detail.

(1) There is a unique spacetime outside spherically symmetric objects.

(2) This spacetime, conveniently represented with the Schwarzschild metric, has a "pit in the potential" compared to Newtonian gravity. That means that, in some sense, gravity is stronger than Newtonian gravity. The difference is negligible far out, but is important closer in.

(3) One of those differences is the presence of an event horizon. Nothing that passes across the event horizon can escape or send anything back to us, including light. Coincidentally, this is at a radius $R = 2GM/c^2$ such that the Newtonian escape velocity is c.

(4) Another difference is that there are unstable circular orbits close enough to a black hole or neutron star. Far away, a slight perturbation of an orbit just makes it a little elliptical. Close enough, however, a slight perturbation causes it to spiral in quickly.

(5) Photons can be deflected dramatically close to a black hole. They can be bent by 2π or more.

(6) Black holes are the simplest macroscopic objects in the universe. Their properties can be described by just their mass, their angular momentum, and their electric charge, and the charge has no important gravitational effect because it is quickly neutralized.

(7) If a black hole rotates, it drags spacetime with it (frame-dragging). This is true of any rotating object, but around a black hole the effects can be dramatic.

(8) There are laws of black holes comparable to the laws of thermodynamics. For example, if two black holes merge then, in analogy to the second law of thermodynamics, the total area of the final single hole is at least as great as the total area of the two initial holes combined.

(9) This thermodynamic analogy extends to black holes actually radiating (Hawking radiation), but the radiation is so slow it is negligible for any stellar-mass black hole (brane theory).

So now, let's think about the spacetime around black holes, and more generally, around anything spherically symmetric.

In Newtonian physics, the gravity around a spherically symmetric object is as if all the mass is concentrated at a point in the center. If the object explodes or contracts, but in a spherically symmetric way, it is still the case that the gravity outside that region is the same as before.

In general relativity, one gets a similar result: if the spacetime is spherically symmetric, then it is unique. Birkhoff's theorem says that the only vacuum, spherically symmetric gravitational field is static. One convenient metric for this spacetime is the Schwarzschild metric:

$$ds^2 = -\left(1-\frac{2M}{r}\right)dt^2 + \left(1-\frac{2M}{r}\right)^{-1}dr^2 + r^2 d\Omega \tag{1}$$

where $d\Omega = d\theta^2 + \sin^2\theta\, d\phi^2$. Remember, here we're setting $G = c = 1$. Note here an important distinction: a *metric* is a description of a spacetime based on a particular choice of coordinates. The spacetime itself is more general. For example, the Minkowski metric describes flat spacetime in one way, but a coordinate transformation makes it look different even though the underlying physics is the same. That's important here, too. The Schwarzschild coordinates are convenient because they are chosen to represent important physical quantities: t is the time between events measured by someone at infinity, θ and ϕ are their usual spherical coordinate equivalents, and r is $1/2\pi$ times the circumference of a circle drawn at Schwarzschild coordinate at r. *However*, you have to be careful: the proper radial distance from r_1 to r_2 is *not* $r_2 - r_1$. It is $\int_{r_1}^{r_2}(g_{rr})^{1/2}\,dr$, which is larger in general. There is an interesting breakdown at $r = 2M$. From looking at the line element (another name for the metric when written in the $ds^2 = \ldots$ form), it appears that extreme badness happens when the radius approaches $2M$. For example, the dr^2 term goes to infinity! But some apparent singularities are just coordinate singularities. An example is the spherical coordinate system: the north or south pole are weird in that system, but nothing unusual happens there. On the other hand, $r = 0$ is a real singularity, where tidal forces are infinite. Nonetheless, $r = 2M$ *is* important, as we'll discuss below.

Ask class: For an observer at a fixed location r, θ, ϕ, how is the proper time interval $d\tau$ measured by that local observer related to the proper time interval dt measured by an observer at infinity for the same two events? Setting $dr = d\theta = d\phi = 0$, we find $d\tau = (-ds^2)^{1/2} = \sqrt{1-2M/r}dt$. **Ask class:** from this, what is the gravitational redshift? What is the time dilation observed? Both are factors of $\sqrt{1-2M/r}$. Distant observers see close observers moving slowly; close observers see distant observers moving fast.

Because of differences in the way in which local or distant observers measure things, in GR one must always be precise when specifying the frame in which a quantity is measured. If necessary, one can then use transformation rules to figure out what that quantity would be in other frames. Confusion between frames is a common sticking-point for many crackpots!

The region $r = 2M$ is the *event horizon*, which defines the "surface" of the black hole and defines the Schwarzschild radius. This is also called the *static limit*: static (non-moving) observers can't exist inside this radius. Now, this also means that if we at infinity watch something fall in to the black hole, it never appears to cross the horizon because of time dilation. It seems to "freeze" there, although it dims rapidly so we shouldn't expect to see a collection of frozen surprised aliens at the horizons of black holes! A person falling through $r = 2M$ does so in finite proper time (one can verify this by integration), and does not feel infinite tidal forces (those happen at $r = 0$, which is a genuine singularity). Note that as $r \longrightarrow 2M$ the redshift approaches infinity. A misunderstanding about what really happens at $r = 2M$ led many people, including Einstein, to disbelieve in black holes because they thought it meant a star somehow could hover just outside the horizon.

The motion of test particles (i.e., those that react to but do not affect the spacetime) depends on whether the particles have zero or nonzero rest mass. Rather than deriving everything, let me just give a few of the important results. These are all for the Schwarzschild spacetime, i.e., without charge or rotation.

For a massive particle:

(1) A local static observer at r measures the velocity of a radially freely falling particle released from rest at infinity as $v^{\hat{r}} = \sqrt{2M/r}$, which is exactly the Newtonian expression!

(2) An observer at infinity measures the orbital frequency of a particle in a circular orbit at Schwarzschild radius r to be $\Omega = \sqrt{GM/r^3}$, which again is exactly the Newtonian expression!

(3) Particles in circular orbits with radii $r < 6M$ are unstable. Let's see if we can figure out a heuristic way of seeing this. **Ask class:** for a small particle orbiting in a circle in Newtonian gravity, how does angular momentum depend on orbital radius? Like $\sqrt{r}$. Suppose we perturb such an orbit by giving it a slightly greater orbital radius at the same angular momentum. **Ask class:** what happens? It falls back, since its angular momentum is too small. Similarly, with a slightly smaller orbital radius it would move out. That is, a perturbation just gives us a circular orbit.

Now, what about GR? **Ask class:** given that GR has "extra strong" gravity, would one expect the angular momentum for a circular orbit a given radius to be larger or smaller than the Newtonian value? Larger, to provide extra support against gravity. In fact, at small radii the required angular momentum increases with decreasing radius. **Ask class:** in the decreasing region of angular momentum, what happens if a particle is perturbed inward? It falls, because the required angular momentum is even more!

This means that there is an innermost stable circular orbit (ISCO), at which the specific angular momentum is $\ell = \sqrt{12}M$ and the specific energy is $e = \sqrt{8/9}$ (we're dividing by m and $c = 1$, so this is $e = \sqrt{8/9}c^2$). Therefore, if gas has spiraled in from infinity in nearly circular orbits, it has released $1 - \sqrt{8/9} \approx 0.057$ of its mass-energy by the time it reaches the ISCO; inside the ISCO it probably releases little energy. The efficiency of energy production is ~ 10 times the efficiency of fusing hydrogen to helium! At the ISCO, a local static observer measures the orbital velocity as $v = c/2$, regardless of the mass of the black hole.

For a massless particle (a photon, to be definite):

(1) If a photon coming from infinity has an impact parameter less than $3\sqrt{3}M$, it will be captured by the black hole. The effective cross section of the black hole for photons is therefore $27M^2$.

(2) Closed orbits are of measure 0: a photon in a circular orbit with a radius $r = 3M$ will continue that way, but it's unstable. A perturbation out means that it escapes, a perturbation in means that it is captured.

The spacetime can also be written exactly, in closed form, if there is rotation. Metrics for this Kerr spacetime (e.g., the Boyer-Lindquist metric) are more complicated than is worth writing here, but it is useful to summarize. The single extra parameter for spinning black holes is a rotational parameter, $a \equiv J/M$. This parameter has the units of mass, and sometimes people use j or $\hat{a} \equiv a/M$ for the dimensionless equivalent. When I mention things like the radius r and the time t, I will use the Boyer-Lindquist values, which reduce to Schwarzschild when $a = 0$ and have similar physical meaning (i.e., t is the time measured by an observer at infinity, and ϕ has the usual meaning of an azimuthal coordinate).

The rotation means that spacetime is no longer spherically symmetric, although it is axisymmetric. The axisymmetry and stationarity (no time dependence) means that the angular momentum and energy are still conserved. The horizon moves into

$$r_+ = M + (M^2 - a^2)^{1/2} . \quad (2)$$

This means that $a < M$ for a black hole to exist (that is, for it to have an event horizon). If somehow it could have $a > M$, there would be no horizon, and would instead be a "naked singularity". A problem that is still unsolved is whether such naked singularities can exist. Nobel Prize winner Roger Penrose's "cosmic censorship conjecture" states that naked singularities are always clothed by horizons, but nothing has been proven.

A new, important radius that crops up is the radius of the "ergosphere". This is at

$$r_0 = M + (M^2 - a^2 \cos^2\theta)^{1/2} . \quad (3)$$

Note that there is room between r_0 and the horizon radius r_+ for $a < 1$ (and even for $a = 1$ it turns out that the horizon and ergosphere are distinct). Inside this radius, test particles (massive or massless) must rotate with the hole, even if they are outside the horizon. The ergosphere is therefore the static limit; things inside can't remain stationary with respect to infinity. This is an extreme example of frame-dragging. Even outside the ergosphere, spacetime is dragged in the direction of rotation of the hole. The natural frequency for this frame-dragging (i.e., the frequency of a particle with no angular momentum) is $\omega \approx \frac{2Ma}{r^3}$. The ergosphere is named thus ("ergo" being Greek for energy) because in principle something sent in there could split in two and emerge with more than the original energy of the particle, thereby tapping the spin energy of the hole. More astrophysically plausible is that threaded magnetic fields down the hole might extract its spin energy (the "Blandford-Znajek process").

In a Kerr spacetime, the location of the ISCO is changed, as is the efficiency of energy extraction. For particles orbiting in the same direction as the hole is spinning, the ISCO moves in; for retrograde particles, the ISCO moves out. For prograde orbits, energy extraction is more efficient if the hole is spinning faster, and the maximum is about 40%(!).

Hawking proved that in any interaction, the surface area of a black hole can never decrease. If many black holes are involved, it's the sum of the areas that can't decrease. This is analogous to the increase of entropy. The area of a Kerr black hole is

$$A = 8\pi M\left[M + (M^2 - a^2)^{1/2}\right] , \tag{4}$$

which reduces to the expected $4\pi(2M)^2$ for $a = 0$. This means, in principle, that two equal-mass $a = M$ black holes with opposite spin could release 50% of their mass-energy in a collision, and two Schwarzschild black holes could release 29%. The actual fraction, based on numerous numerical simulations, is a few percent and depends on the mass ratio and the spins of the two black holes.

Jacob Bekenstein tried to extend this thermodynamic analogy, but Hawking got the big prize: black holes have an effective temperature, and radiate! The temperature is

$$T = \frac{\hbar}{8\pi kGM/c^3} = 10^{-7}\mathrm{K}\left(\frac{M_\odot}{M}\right) . \tag{5}$$

This is essentially because virtual particles can be separated by the gravitational field and one can be made real (analogous to pair creation in an electric field). It is therefore a remarkable marriage of quantum mechanics and general relativity. However, the timescale of emission is ridiculously long (around $10^{67}(M/M_\odot)^3$ yr), so only mini black holes with mountain-type masses of 10^{15} g would evaporate in the current age of the universe. There is no evidence for such things existing, although conceivably they might have been formed near the beginning of the universe, and they are what we might consider a dark horse candidate for dark matter. This is therefore not likely to be tested in the foreseeable future, especially because radiation from other sources (e.g., the accretion of stray molecules onto black holes) dwarfs Hawking radiation.

Astrophysical applications of GR and black holes

Hydrostatic equilibrium applies just as well in GR as in Newtonian physics, but the gravity is different and therefore the specific equation is different (the Oppenheimer-Volkoff equation; see Wikipedia or many other references.).

But we shouldn't exit our lecture without saying a few words about whether black holes actually exist! Their identification poses a difficult problem, because their existence is something of a negative proposition: if nothing else fits, maybe it's a black hole. However, good evidence is to be had from a variety of quarters. Neutron stars can't have $M > 3\,M_\odot$, so if a compact X-ray source in a binary is seen that has a mass greater than this, it's probably a black hole. On a larger scale, dynamical measurements of stellar movements in the cores of several galaxies imply $M > 10^8\,M_\odot$ or so in small region ($\sim$pc). This, too, probably means there are black holes there, although one has to be careful; there was once an announcement of a $10^{11}\,M_\odot$ discovered in this fashion, but it turned out that the core was actually the location of a galactic collision, so the velocities weren't orbital velocities. Gravitational wave detections are fully consistent with the expectations from GR and black holes. Recently, the Event Horizon Telescope have shown that there are "shadows" of the expected sizes from two supermassive black holes (in our Galactic center and in the center of the galaxy M87), although they have *not* demonstrated the existence of an event horizon.

In any case, there is now virtually no reasonable opposition to the existence of black holes, although there are still those who have philosophical objections to the ideas of event horizons. Also, more seriously, we still need to investigate the possibility that gravity isn't quite described by GR, or that there are other objects, e.g., ones without horizons, which are "black hole mimickers". In science, after all, we should never say that we have the final answer; that's what makes our work fun!