The Sun: A Great Big Heap a' Friendly Self-Luminescent Gas

The Sun, being so close, allows us a far more detailed opportunity to study it than most stars allow. This means we can learn a great deal about the Sun that isn't available otherwise, and hence detailed aspects of stellar structure and evolution. **Ask class:** what are some of the things we can observe only on the Sun? Examples include neutrino fluxes, extremely detailed helioseismology, and angular resolution of sunspots (allowing observations of rotation). We will consider some of the information available via these paths.

Much of the evolution of the Sun follows the low-mass path we discussed earlier. As in the last class, if we wanted to do a detailed model, we would construct it from the basic equations of stellar structure, with the known mass and composition of the Sun, then proceed to evolve it under conditions of near-equilibrium. But unless we want to become specialists in this field, we get more insight by scaling techniques.

Let's start with a uniform density star supported by ideal gas pressure. The internal energy density is

$$E = \frac{3}{2}nkT = \frac{3}{2}\rho\frac{kT}{\mu m_p}\ . \tag{1}$$

The internal energy $U = EV$, and since $\rho V = M$, $U = \frac{3}{2}MkT/(\mu m_p)$. The virial theorem implies that $U = -\Omega/2$, which for a uniform density star is $\frac{3}{10}GM^2/R$. Equating, we find

$$T \approx 4\times 10^6\mu\left(\frac{M}{M_\odot}\right)^{2/3}\rho^{1/3}\ \mathrm{K}\ . \tag{2}$$

You may reasonably object that the constant density approximation surely doesn't hold for the Sun, and aren't we about ready to give up these silly approximations? Well, sort of. **Ask class:** what can one say about how T should depend on M and R, from dimensional arguments and the virial theorem? Since kT is an energy, one could say that it should be related to the kinetic energy per particle, which is $\sim GMm/R \sim M/R$, so $T \sim M/R$. Expressing this in terms of M and $\rho \sim M/R^3$, one can say generally that $T \propto \mu M^{2/3}\rho^{1/3}$ is the scaling one expects from a large variety of models. So we'll go with that for the moment.

Comment: the following approach is simply to get a general idea of how the Sun's luminosity changes with time. Therefore, in order to get a quick estimate, we are going to make a number of approximations. These include dropping all constant factors, and using ratios instead of derivatives. This will give the overall view of the evolution: does the Sun become brighter or dimmer with time, and by roughly how much? The constant factors could be dropped anyway, because part of the scaling solution strategy is to normalize at the end using observations (e.g., the current luminosity of the Sun).

Slipping further into the debauchery of scaling approximations, we can take the energy transfer equation

$$L = -\frac{4\pi r^2 c}{3\kappa\rho}\frac{d(aT^4)}{dr} \tag{3}$$

and get some kind of average over the star by "cancelling the d's" and writing $L \propto RT^4/\kappa\rho$. Assuming a Kramers opacity, $\kappa = \kappa_0\rho T^{-3.5}$, and using the mass equation $R \propto (M/\rho)^{1/3}$, we find

$$L \propto \frac{M^{5.33}\rho^{0.117}\mu^{7.5}}{\kappa_0} \ . \tag{4}$$

From this, the most important factor in the change in the Sun's luminosity as it evolves is the mean molecular weight, because the mass changes little (winds are negligible on the main sequence). **Ask class:** qualitatively, from this would one expect the Sun's luminosity to go up or down as it evolves? Up, because conversion of hydrogen to helium increases the mean molecular weight.

How else can we simplify? **Ask class:** how much do they trust the exact exponents written above? Not much, given the other approximations. So what can we do? First, the mass of the Sun hardly changes at all during its main sequence lifetime (the solar wind spits out $\dot{M} \approx 10^{-14}\, M_\odot\ \mathrm{yr}^{-1}$), so we can ignore that factor. What about other factors? **Ask class:** how much do they expect that the Sun's luminosity has changed over the past four billion years? Not too much, otherwise life couldn't have arisen (if the Earth were frozen or boiled). We also know in advance that the radius hasn't changed much, so neither did ρ, and with a small exponent we can drop that factor as well. How about κ_0? **Ask class:** in the majority of the Sun, do they expect κ_0 to depend significantly on the hydrogen and helium fractions, given that bound-free dominates? No, it's mainly metals. So we pretend that κ_0 is more or less constant with time as well and end up with

$$\frac{L(t)}{L(0)} = \left[\frac{\mu(t)}{\mu(0)}\right]^{7.5} \ . \tag{5}$$

We therefore need to figure out how μ changes with time. We know that $\mu(t) = 4/(3+5X(t))$, so it's a matter of how rapidly the hydrogen is consumed. **Ask class:** If there is a luminosity $L(t)$ generated by hydrogen burning, and a release of $Q = 6\times10^{18}$ erg g^{-1}, what is the change in $X(t)$ with time for a star of mass M? It's $dX/dt = -L/MQ$. We then find

$$\frac{d\mu(t)}{dt} = \frac{5}{4}\mu^2(t)\frac{L(t)}{MQ} \ . \tag{6}$$

The differential equation for $L(t)$ becomes

$$\frac{dL(t)}{dt} = \frac{75}{8}\frac{\mu(0)}{MQ}\frac{L^{32/15}(t)}{L^{2/15}(0)} \ , \tag{7}$$

with the solution

$$L(t) = L(0)\left[1 - \frac{85}{8}\frac{\mu(0)L(0)}{MQ}t\right]^{-15/17} \ , \tag{8}$$

or for the Sun, with $t_\odot = 4.5 \times 10^9$ yr and $\mu(0) \approx 0.6$,

$$\frac{L(t)}{L_\odot} \approx \frac{L(0)}{L_\odot}\left[1 - 0.3\frac{L(0)}{L_\odot}\frac{t}{t_\odot}\right]^{-15/17} \quad . \tag{9}$$

Fitting $L(t_\odot) = L_\odot$, $L(0) \approx 0.8\, L_\odot$. Now, incidentally, given our other crimes against exact solution, I personally would have made the approximation $32/15 \approx 2$ and $2/15 \approx 0$ in the differential equation for $L(t)$, so the final solution would have a -1 instead of a $-15/17$, but that's a personal preference. So, it appears that the Sun initially had a luminosity about 20-30% less than it does now. We discussed some of the implications of the "faint young Sun paradox" in a previous class, but now we can ask: what does this say about the conditions necessary for life elsewhere, if we assume that surface liquid water is necessary?

Rotation

Now we need to take a hesitant step away from spherical symmetry and consider rotation. Be warned that for this we'll need to delve into considerably more complicated equations even in the simplest approximation. Try to avoid whiplash!

We know that stars rotate. **Ask class:** what modifications have to be made? One consequence is that the equation of hydrostatic equilibrium has to be modified to take centrifugal acceleration into account. We also can't just use dP/dr, since with rotation we no longer have spherical symmetry. Thus we need to use the vector gradient, ∇P, instead of just dP/dr. Then we have $\nabla P = -\rho g_{\rm eff}$, where $g_{\rm eff}$ includes both gravity and centrifugal terms. **Ask class:** can anything fundamental about the star depend on the angular velocity ω to the first power? No, because that would mean that the *direction* of rotation matters, which it can't (imagine standing on your head and looking at the star; the direction has changed, but obviously the structure didn't). The centrifugal acceleration for angular velocity ω a distance s from the rotation axis is $\omega^2 s\mathbf{e}_s$, where $\mathbf{e}_s$ is a unit vector perpendicular to the rotation axis, pointing outwards. Things are especially simple if the centrifugal acceleration can be derived from a potential:

$$\omega^2 s\mathbf{e}_s = -\nabla V \ . \tag{10}$$

This is equivalent to saying that the angular velocity depends only on s, so that it is constant on cylinders. This is called conservative rotation.

We can then combine the gravitational potential ϕ (so that the acceleration of gravity is $-\nabla\phi$) and centrifugal potential V: $\Psi = \phi + V$ (where $V = -\int_0^s \omega^2 s\, ds$) and rewrite the equation of hydrostatic equilibrium as

$$\nabla P = -\rho\nabla\Psi \ . \tag{11}$$

This means that the pressure gradient and potential gradient surfaces are parallel to each other. Therefore, equipotential surfaces Ψ=constant correspond to surfaces of constant pressure, so pressure is a function of Ψ only. We can take the curl of the equation of hydrostatic equilibrium to find $0 = -\nabla\psi \times \nabla\rho$ (remember that the curl of a gradient is zero, so $\nabla \times \nabla P = 0$, for example), showing that the density is also constant on equipotential surfaces. If we assume an ideal gas, in which $P = (\rho/\mu m_p)kT$, then ρT is also only a function of Ψ if the star is chemically homogeneous; since ρ is only a function of ψ, this means that for a chemically homogeneous star T is constant on equipotential surfaces. If the star is not chemically homogeneous, then it is T/μ that is constant on equipotentials. We can also see that if the pressure is a function only of the density, $P(\rho)$, then the rotation must be conservative, because $\nabla P(\rho)/\rho$ can be expressed as the gradient of a potential:

$$\frac{1}{\rho}\frac{dP}{dr} = \frac{1}{\rho}\frac{dP}{d\rho}\frac{d\rho}{dr} = -\frac{d\Psi}{d\rho}\frac{d\rho}{dr} , \tag{12}$$

so $\Psi = -\int \frac{dP}{d\rho}(1/\rho)d\rho$. This implies that the rotation of white dwarfs should be conservative, and hence equal along cylinders, because the pressure is degeneracy pressure and therefore depends on density alone (to a good approximation).

Now let's consider what happens when we include radiative energy transport. The equation of transport can be written

$$\mathbf{F} = -\frac{c}{3\kappa\rho}\nabla(aT^4) = -\frac{c}{3\kappa\rho}\frac{d(aT^4)}{d\Psi}\nabla\Psi = -k(\Psi)\nabla\Psi , \tag{13}$$

(again using ∇ because we can't assume spherical symmetry any more) because $\kappa = \kappa(\rho, T) = \kappa(\Psi)$. Here again $-\nabla\Psi$ is the effective gravity including centrifugal terms. Now combine this with the equation of energy conservation, which we write in the form $\nabla \cdot \mathbf{F} = \epsilon\rho$:

$$\nabla \cdot \mathbf{F} = -\frac{dk}{d\Psi}(\nabla\Psi)^2 - k(\Psi)\nabla^2\Psi = -\frac{dk}{d\Psi}(\nabla\Psi)^2 - k(\Psi)\left(4\pi G\rho - \frac{1}{s}\frac{d(s^2\omega^2)}{ds}\right) = \epsilon\rho . \tag{14}$$

Here we've made use of Poisson's equation, $\nabla^2\Phi = 4\pi G\rho$. Can this equation hold everywhere along equipotentials? No! Everything except the first term and the second term in parentheses is a function of Ψ only (remember $\epsilon = \epsilon(\rho, T) = \epsilon(\Psi)$), and is therefore constant along equipotentials. Consider now as an example something that rotates uniformly; then $(1/s)d(s^2\omega^2)/ds$ is constant, period. However, the first term, $-dk/d\Psi(g_{\rm eff})^2$, is definitely *not* constant; the effective gravity is greater at the poles than at the equator, along equipotentials.

This result, that when rotation enters that radiative transport and simple energy conservation can't be satisfied simultaneously, is known as von Zeipel's paradox. The way out comes from the realization that we have overconstrained the problem; when rotation exists, another form of energy transport arises. This is meridional circulation. Note, by the way, that the nonconservation of energy under our strict assumptions goes like ω^2, as was expected physically. Meridional circulation also happens for non-conservative rotation. In a radiatively dominated region of the star (specifically where $(\nabla_{\rm ad} - \nabla)/\nabla_{\rm ad} \sim 1$), and assuming that the density in the region of interest is close to the mean density of the star, then the circulation time scale is approximately the *Eddington-Sweet* time scale

$$\tau_{\rm circ} \approx \frac{GM^2}{LR}\frac{1}{\chi} \approx \frac{\tau_{\rm KH}}{\chi} \ , \tag{15}$$

where $\chi \equiv \frac{\omega^2}{2\pi G\rho_c} \approx (\omega/\omega_K)^2$ is a parameter describing the importance of rotation. For the Sun, $\chi \approx 10^{-5}$, and since $\tau_{\rm KH} \approx 10^7$ yr, circulation is not important in the deep envelope. However, near the surface, the time scale is shortened by a factor $\approx \rho/\bar{\rho}$, which can be tiny in the outer envelope. Meridional circulation is therefore important there.