Basics of Radiation Fields

Initial questions: How could you estimate the distance to a radio source in our galaxy if you don't have a parallax?

We are now going to shift gears a bit. In order to understand better what light is and what can be measured about it, we'll get down to the basics of electromagnetism. First, though, some historical context.

A lot of mechanical things are easily observed. One can see a ball fall, things bounce off each other, planets move, and so on, without too much effort (although quantitative measurement requires care). Electromagnetism is a different story. Although electromagnetic interactions (or more properly quantum electrodynamics) account for all nongravitational phenomena we can easily see, it's tougher to distill it down to something palpable. As a result, it wasn't until the 19th century that people started to isolate some of the quantitative laws governing electricity and magnetism. As you know, the crowning achievement in this period came when Maxwell produced the fundamental equations of electromagnetism. In cgs units:

$$\begin{array}{llll} \nabla\cdot\mathbf{D} & = 4\pi\rho & \nabla\cdot\mathbf{B} & = 0 \\ \nabla\times\mathbf{E} & = -\frac{1}{c}\frac{\partial\mathbf{B}}{\partial t} & \nabla\times\mathbf{H} & = \frac{1}{c}\frac{\partial\mathbf{D}}{\partial t}+\frac{4\pi}{c}\mathbf{j} \end{array} \tag{1}$$

where $D=\epsilon E$ and $H=B/\mu$, with ϵ the dielectric constant and μ the magnetic permeability. In a vacuum, $\epsilon=\mu=1$.

This is an amazing accomplishment. When combined with the Lorentz force law

$$\mathbf{F}=q[\mathbf{E}+(1/c)\mathbf{v}\times\mathbf{B}]\ , \tag{2}$$

Maxwell's equations encompass the entirety of classical electromagnetism. It is difficult to overstate the magnitude of this advance. This was Einstein's starting point for special relativity, it has motivated a lot of mathematical physics, and the structure of the equations themselves have provided a guide to the development of many subsequent (and unrelated) physical theories. In order to gain insight about the equations and the properties of light, we need to manipulate the equations in a few ways.

Ask class: how would the equations be changed if there were magnetic monopoles? There would be source terms for the magnetic field, so the divergence would be $4\pi\rho_B$ instead of 0, and there would be a magnetic current as well. The lack of measured deviation from Maxwell's equations as they stand indicates that magnetic monopoles aren't common, but there is still the possibility that some exist. As an aside, one of the motivations for inflationary cosmology was to explain the absence of magnetic monopoles (in prior cosmologies, they were expected to be ubiquitous).

A great deal of insight into Maxwell's equations can be obtained by taking curls or divergences, and seeing what results. For example, if you take the divergence of the fourth equation and combine it with the time derivative of the first you get

$$\nabla \cdot \mathbf{j} + \frac{\partial \rho}{\partial t} = 0 \; . \tag{3}$$

This is a classic conservation equation, of charge in this case. It says that the change in the total charge in a given volume is the negative of the divergence of the current; that is, the only change in charge comes from charge flowing in or out, rather than actual creation or destruction of charge. Note that this is also true relativistically: creation or destruction of an electron-positron pair doesn't change the total charge. As a personal comment, I think it may be revealing that as far as we can tell the total charge of the universe is identically zero. It suggests to me that even at the Planck time, charge conservation applied. You can contrast this with the symmetry between matter and antimatter. At current laboratory energies it's impossible to create antimatter without creating matter, but the universe is almost all matter so there had to be some asymmetry early on.

One last thing before we look into the propagation of waves. In mechanics it is often helpful to think of a potential instead of a field (for example, the gravitational force is $\mathbf{F} = -\nabla\phi$). Can we do the same thing with electromagnetism? Since $\nabla \cdot \mathbf{B} = 0$, we can write $\mathbf{B} = \nabla \times \mathbf{A}$, where $\mathbf{A}$ is a vector potential. The equation for $\nabla \times \mathbf{E}$ can then be written

$$\nabla \times \left(\mathbf{E} + \frac{1}{c} \partial \mathbf{A}/\partial t \right) = 0 \; , \tag{4}$$

so the () in this equation can be written as the gradient of a scalar potential $-\phi$ (where the negative sign is conventional), meaning that

$$\mathbf{E} = -\nabla\phi - \frac{1}{c}\partial\mathbf{A}/\partial t \; . \tag{5}$$

This means that, in principle, you can use the potentials instead of the fields. Now, **Ask class:** suppose that you have chosen $\mathbf{A}$ and ϕ, and then make the following substitutions:

$$\begin{array}{ll} \mathbf{A} & \rightarrow \mathbf{A} + \nabla\psi \\ \phi & \rightarrow \phi - \frac{1}{c}\partial\psi/\partial t \; , \end{array} \tag{6}$$

where ψ is *any* scalar function? What happens to $\mathbf{B}$ and $\mathbf{E}$? Nothing! This means that for a given $\mathbf{E}$ and $\mathbf{B}$, the potentials are *not* uniquely defined. However, since all physical observables come from the fields, this means that the theory doesn't care about this arbitrariness in the potentials. This is called *gauge invariance.* It allows a choice of potentials to simplify the equations, but more generally also is a powerful guide to the construction of physical theories.

Note that the potential and field formulations are completely identical. Indeed, most of the complexity (and richness) you'll find in E&M textbooks comes from various ways of looking at Maxwell's equations and cute tricks to solve particular problems. In principle, "all" you have to do to solve any electromagnetism problem is cram Maxwell's equations and the Lorentz force law into a computer, then grind away. In practice, of course, you get a lot more insight the old-fashioned way, but the underlying simplicity shouldn't get lost in the haze.

Now let's move on to radiation in a vacuum. **Ask class:** what are Maxwell's equations in a vacuum? Remember that $\epsilon = \mu = 1$.

$$\begin{aligned} \nabla \cdot \mathbf{E} &= 0 & \nabla \cdot \mathbf{B} &= 0 \\ \nabla \times \mathbf{E} &= -\tfrac{1}{c}\partial \mathbf{B}/\partial t & \nabla \times \mathbf{B} &= \tfrac{1}{c}\partial \mathbf{E}/\partial t \; . \end{aligned} \tag{7}$$

The equations look a lot more symmetric now; in fact, you get them back again if you substitute $\mathbf{E} \rightarrow \mathbf{B}$, $\mathbf{B} \rightarrow -\mathbf{E}$.

Let's take the curl of the third equation. If we use the vector identity $\nabla \times (\nabla \times \mathbf{E}) = \nabla(\nabla \cdot \mathbf{E}) - \nabla^2 \mathbf{E} = -\nabla^2 \mathbf{E}$ in this case, we get

$$\nabla^2 \mathbf{E} - \frac{1}{c^2}\partial^2 \mathbf{E}/\partial t^2 = 0 \; . \tag{8}$$

This is a classic wave equation. You'd get the same thing for $\mathbf{B}$. What this means is that electromagnetic waves propagate at the speed of light c. Note in particular that this is a completely general statement, independent of the frequency of the light (as long as it's in vacuum).

We can get further insight by adopting a technique with incredibly general applications. Any arbitrary function can be decomposed into sums (or integrals) of plane waves. Therefore, if we find out what happens to plane waves, we can extend this to more general functions.

We thus consider solutions of the form

$$\begin{aligned} \mathbf{E} &= \hat{\mathbf{a}}_1 E_0 \exp i(\mathbf{k} \cdot \mathbf{r} - \omega t) \\ \mathbf{B} &= \hat{\mathbf{a}}_2 B_0 \exp i(\mathbf{k} \cdot \mathbf{r} - \omega t) \end{aligned} \tag{9}$$

Here E_0 and B_0 are complex constants, and $\hat{\mathbf{a}}_1$ and $\hat{\mathbf{a}}_2$ are unit vectors. Here $\mathbf{k} = k\mathbf{n}$ is the wave vector (units cm^{-1}) and ω is the frequency. k itself is called the wavenumber. These solutions represent waves traveling in the $\mathbf{n}$ direction (note that solutions of constant phase, i.e., the argument of the exponential, move in that direction).

Before substituting this into Maxwell's equations, we should think about what we expect. **Ask class:** are light waves transverse (oscillating perpendicular to the direction of motion) or longitudinal (oscillating along the direction of motion) or a combination? They are transverse, so that's what we should get from this substitution. We find:

$$\begin{aligned} i\mathbf{k}\cdot\hat{\mathbf{a}}_1 E_0 &= 0 & \qquad i\mathbf{k}\cdot\hat{\mathbf{a}}_2 B_0 &= 0 \\ i\mathbf{k}\times\hat{\mathbf{a}}_1 E_0 &= (i\omega/c)\hat{\mathbf{a}}_2 B_0 & \qquad i\mathbf{k}\times\hat{\mathbf{a}}_2 B_0 &= -(i\omega/c)\hat{\mathbf{a}}_1 E_0\ . \end{aligned} \tag{10}$$

The first two equations tell us, as we expected, that the electric and magnetic fields are perpendicular to the direction of propagation. **Ask class:** what do the second two equations say about the relative directions of $\mathbf{E}$ and $\mathbf{B}$? They are perpendicular. This means that $\mathbf{k}$, $\hat{\mathbf{a}}_1$, and $\hat{\mathbf{a}}_2$ form three orthogonal axes.

Our next bit of understanding comes from solving for E_0 and B_0. Since $\mathbf{k}$ is perpendicular to $\hat{\mathbf{a}}_1$, the magnitude of $\mathbf{k}\times\hat{\mathbf{a}}_2$ is just k (because $\hat{\mathbf{a}}_1$ is a unit vector). Therefore, we have

$$E_0 = (\omega/kc)B_0, \qquad B_0 = (\omega/kc)E_0\ . \tag{11}$$

Therefore $E_0 = (\omega/kc)^2 E_0$, or $\omega^2 = c^2k^2$. This means $E_0 = B_0$. Taking the positive root, $\omega = kc$. This means that the phase velocity is $\omega/k = c$ and the group velocity is $d\omega/dk = c$. A relation between ω and k is called a *dispersion relation*. This particular relation is *nondispersive*, because there is no dependence of the velocities on the wavenumber. As a result, a wave packet will stay together; if there were a dependence of velocity on wavenumber, the packet would spread. Of course, since we're considering propagation in a vacuum, this all comes from the knowledge that all frequencies of light propagate with the speed of light in a vacuum. But this is a useful alternative way of looking at it, and when there is matter present and $\epsilon \neq 1$, $\mu \neq 1$, dispersion occurs.

Now let's think about the flux of energy. Energy is conserved. That means that if you pick some fixed volume in space, the rate of change in its total energy density must be equal to negative the divergence of its flux (this is exactly the form we got for conservation of charge). However, in order to determine the form of the energy conservation law in our case, we need to determine what the energy density and flux are.

Rewriting the Lorentz force law as a force per unit volume,

$$\mathbf{f} = \rho\mathbf{E} + \frac{1}{c}\mathbf{j}\times\mathbf{B}\ , \tag{12}$$

where ρ and $\mathbf{j}$ are respectively the charge and current densities. The magnetic force operates perpendicularly to the velocity, so it can't do any work. Therefore, the rate of change of mechanical energy per unit volume caused by the action of the electromagnetic fields is

$$dU_{\rm mech}/dt = \mathbf{j}\cdot\mathbf{E}\ . \tag{13}$$

We can use Maxwell's equations to relate $\mathbf{j}$ to $\mathbf{H}$ and $\mathbf{D}$ (note that for the moment we're going back to the general definition. We'll return to the vacuum case shortly.). If we do this and manipulate it a bit (see page 53 of the book), we find

$$\mathbf{j}\cdot\mathbf{E} + \frac{1}{8\pi}\frac{\partial}{\partial t}\left(\epsilon E^2 + B^2/\mu\right) = -\nabla\cdot\left(\frac{c}{4\pi}\mathbf{E}\times\mathbf{H}\right)\ . \tag{14}$$

Ask class: given our energy conservation arguments, what are each of these terms? This is exactly the form we need: the rate of change in energy density equals negative the divergence of a flux. Since we already identified $\mathbf{j}\cdot\mathbf{E}$ as the rate of change of mechanical energy density, the second term on the left indicates the energy density of the electromagnetic field:

$$U_{\rm field} = \frac{1}{8\pi}(\epsilon E^2 + B^2/\mu)\ . \tag{15}$$

This naturally separates into a contribution by the electric field and one by the magnetic field. The other identification we have is that the energy flux vector must be the Poynting vector

$$\mathbf{S} = (c/4\pi)\mathbf{E}\times\mathbf{H}\ . \tag{16}$$

Let's now return to propagation in a vacuum. We'd like to figure out the energy density and Poynting flux, but since $\mathbf{E}$ and $\mathbf{B}$ both vary sinusoidally in time, the energy density and Poynting flux will as well. The most representative single number is a time average. As shown in the book (pg. 57), the time-averaged Poynting flux is $\langle S\rangle = (c/8\pi)|E_0|^2 = (c/8\pi)|B_0|^2$ and the time-averaged energy density is $\langle U\rangle = (1/8\pi)|E_0|^2 = (1/8\pi)|B_0|^2$. **Ask class:** from dimensional arguments, how do we get the velocity of energy flow from $\langle S\rangle$ and $\langle U\rangle$? You can always get a flux from an energy density by multiplying by a speed, so the velocity of energy flow is $\langle S\rangle/\langle U\rangle = c$. Sure enough, the energy flow propagates at the speed of light, as we might have expected.

The last point in this lecture will be about the radiation spectrum. When you think about measuring a spectrum (say, optically), you usually have in mind intensity as a function of wavelength or frequency. In reality, though, at some fundamental level what you're doing is measuring the time variation of the electric field (and the magnetic field, but that just mirrors what the $\mathbf{E}$ field is doing). Going from the time domain to the frequency domain tells us we need to take a Fourier transform. It also indicates that we have limits on how well we can resolve the frequency; for example, if we observe for a time Δt, then the frequency resolution (defined in some sense) is $\Delta\omega > 1/\Delta t$. This mimics the uncertainty relation in quantum mechanics, but is not directly quantum mechanical in origin. With this in mind, the frequency spectrum is

$$\hat{E}(\omega) = (1/2\pi)\int_{-\infty}^{\infty} E(t)e^{i\omega t}dt\ , \tag{17}$$

with the inverse transform

$$E(t) = \int_{-\infty}^{\infty} \hat{E}(\omega)e^{-i\omega t}d\omega\ , \tag{18}$$

Incidentally, I've never understood why this isn't written symmetrically, with factors of $1/\sqrt{2\pi}$ on both sides. Convention, I guess. If there is a pulse of electromagnetic radiation of some duration, or if the electromagnetic field is measured for some finite time, the energy per area per frequency interval is

$$\frac{dE}{dA\,d\omega} = c|\hat{E}(\omega)|^2 \ . \tag{19}$$

Note, therefore, that the frequency resolution and sharpness of the sharpest features in the spectrum are limited by the duration of the pulse or observation. This is rarely a limit for astronomical observations. **Ask class:** for meter wavelength radio waves, what is the maximum possible resolution in a one second observation? The frequency of the waves is 3×10^8 Hz, so over one second the resolution could be 3×10^8. Other factors interfere long before this fundamental limit.

Recommended Rybicki and Lightman problem: 2.4