Bremsstrahlung

Initial questions: How does the hot gas in galaxy clusters cool? What should we see in their inner portions, where the density is high?

As in the last lecture, we're going to take a more in-depth look at a particular radiative process, in this case bremsstrahlung. The term means "braking radiation", and occurs when an electron is accelerated by passage near an ion, and hence radiates. There is also an inverse process, which you get by "running the film backward in time": a photon is absorbed by an electron that is moving in the Coulomb field of an ion. This is free-free absorption.

Bremsstrahlung and free-free absorption are basic radiative processes that show up in many contexts. Given our limited time, we will only be able to touch on a few of the concepts; read Rybicki and Lightman or Shu if you want more details.

Let's make things easier for ourselves by starting with nonrelativistic bremsstrahlung. From our discussion of acceleration radiation, this means we'll use a dipole approximation. In such a case, this means the radiation is proportional to the second derivative of the dipole moment $\sum e_i \mathbf{r}_i$. Note that for interactions of identical particles (all electrons, or all protons, for example), the dipole moment is proportional to $\sum m_i \mathbf{r}_i$, which is the center of mass. **Ask class:** what can we conclude about bremsstrahlung from identical particles? There is no dipole radiation (although there can be quadrupole radiation), because the center of mass is stationary. We thus consider electron-ion bremsstrahlung. In principle, both are moving, so we have to include acceleration radiation from both electrons and ions. **Ask class:** is there a simplification that can help us out? Since ions are so much more massive than electrons, to a good approximation we can treat them as fixed, while the electrons move in their static Coulomb field. That's an approximation that appears over and over again in plasma physics.

As with Compton scattering, we'll simplify things further by considering single-speed electrons at the start. As with lots of these basic processes, one can "simply" grind through with brute force, but in this case there are several important obstacles to overcome, so we'll go into this in more detail. First, let's assume that the electron is only slightly deflected from its path; that means that instead of having to compute the full trajectory, we can assume a straight line path and determine the radiation emitted as a result.

Suppose the electron has charge $-e$, and the ion has charge Ze. The impact parameter is b; that means that if the path were a perfect straight line, the closest the electron would come to the ion would be a distance b. The dipole moment is $\mathbf{d} = -e\mathbf{R}$, where $\mathbf{R}$ is the instantaneous location of the electron, so the second derivative is $\ddot{\mathbf{d}} = -e\dot{\mathbf{v}}$, where $\mathbf{v}$ is the electron velocity. **Ask class:** schematically, how would they compute the total energy radiated over the electron's trajectory?

More generally, however, we would like to figure out the energy radiated as a function of frequency. Here's a case where Fourier transforms can add some insight. Recall that the Fourier transform of a quantity, in this case $\mathbf{d}$, is

$$\hat{\mathbf{d}} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathbf{d}(t)e^{i\omega t}dt \ . \tag{1}$$

Thus the Fourier transform of $\ddot{\mathbf{d}}$ at some frequency ω is $-\omega^2\hat{\mathbf{d}}(\omega)$ (think of taking the derivative inside the integral, where it acts on $\exp(i\omega t)$). This gives

$$-\omega^2\hat{\mathbf{d}}(\omega) = -(e/2\pi)\int_{-\infty}^{\infty} \dot{\mathbf{v}}e^{i\omega t}\,dt \ . \tag{2}$$

In general this still might be a mess to evaluate, but let's get some insight by looking at extreme cases: large and small ω. We know that for most of the trajectory of the electron it's far from the ion, but it is close to its minimum distance over a time called the *collision time*, $\tau = b/v$. That means the integral above is only really important for times between $-\tau/2$ and $\tau/2$, give or take. Now, if $\omega\tau \gg 1$, then the exponential oscillates many times, so the net contribution is small. If instead $\omega\tau \ll 1$, then the exponent is close to 0 and thus the exponential is near unity. Thus, in our two limits:

$$\begin{aligned} \hat{\mathbf{d}}(\omega) \ &\sim (e/2\pi\omega^2)\boldsymbol{\Delta}\mathbf{v}, \qquad \omega\tau \ll 1 \\ &\sim 0, \qquad \omega\tau \gg 1 \ . \end{aligned} \tag{3}$$

Here $\boldsymbol{\Delta}\mathbf{v}$ is the change of velocity during the collision.

Now let's think about how we could get this to a form where we have the energy emitted per frequency interval. We know that the energy per area per time is given by the Poynting flux

$$\frac{dW}{dtdA} = \frac{c}{4\pi}E^2(t) \ . \tag{4}$$

Thus the energy per area is this, integrated over time: $dW/dA = (c/4\pi)\int_{-\infty}^{\infty} E^2(t)dt$. Parseval's theorem says that the integral over all times of a quantity is related to the integral over all frequencies of the quantity's Fourier transform via

$$\int_{-\infty}^{\infty} E^2(t)dt = 2\pi\int_{-\infty}^{\infty} |\hat{E}(\omega)|^2 d\omega \ . \tag{5}$$

Here the vertical bars mean the magnitude of the complex quantity $\hat{E}(\omega)$. E is real, which tells us that $|\hat{E}(\omega)|^2 = |\hat{E}(-\omega)|^2$, so this gives

$$\begin{aligned} dW/dA \ &= c\textstyle\int_0^{\infty} |\hat{E}(\omega)|^2 d\omega \\ dW/dAd\omega \ &= c|\hat{E}(\omega)|^2 \ . \end{aligned} \tag{6}$$

Back when we discussed acceleration and radiation, we found that

$$E(t) = \ddot{d}(t)\frac{\sin\Theta}{c^2 R_0} \tag{7}$$

for an angle Θ relative to the polarization direction and an average distance R_0 from the charges. Integrating over solid angle, and realizing that $dA = R_0^2 d\Omega$, we get

$$\frac{dW}{d\omega} = \frac{8\pi\omega^4}{3c^3} |\hat{d}(\omega)|^2 \ . \tag{8}$$

Plugging this into our bremsstrahlung formula, this means that the energy emitted during the whole encounter in a frequency interval $d\omega$ is

$$\begin{aligned} dW/d\omega \ &\approx (2e^2/3\pi c^3)|\mathbf{\Delta v}|^2, \qquad \omega\tau \ll 1 \\ &\approx 0, \qquad \omega\tau \gg 1 \ . \end{aligned} \tag{9}$$

But how do we estimate $\mathbf{\Delta v}$? Most of the change will be deflecting the electron, rather than speeding it up or slowing it down. That means we can consider just the change in velocity normal to the path:

$$\Delta v = \frac{Ze^2}{m} \int_{-\infty}^{\infty} \frac{b dt}{(b^2 + v^2 t^2)^{3/2}} = \frac{2Ze^2}{mbv} \ . \tag{10}$$

Putting it together, this means the energy radiated in a frequency interval $d\omega$, for electrons with impact parameter b, is

$$\begin{aligned} dW(b)/d\omega \ &= [8Z^2 e^6/(3\pi c^3 m^2 v^2 b^2)], \qquad b \ll v/\omega \\ &= 0, \qquad b \gg v/\omega \ . \end{aligned} \tag{11}$$

Here $b \gg v/\omega$ is a restatement of $\omega\tau \gg 1$. Now wait a second: isn't there a problem? This says that the power per frequency emitted for extremely small b diverges. Hm. **Ask class:** any ideas at this point what we can do? If not, let's plow on anyway.

If we have lots of electrons (say, number density n_e) moving with a fixed speed v, then the flux of electrons (number per area per time) is $n_e v$. The number per time that have an impact parameter between b and $b + db$ is this flux times the area element, $2\pi b\, db$. If the number density of ions is n_i, this means that the energy per frequency per volume per time is

$$\frac{dW}{d\omega dV dt} = n_e n_i 2\pi v \int_0^{\infty} \frac{dW(b)}{d\omega} b\, db \ . \tag{12}$$

Yikes! Danger, Will Robinson! The integral in question is of $1/b$, so we get a logarithm that diverges at *both* large b and small b. **Ask class:** what do we do? This is one manifestation of what we discussed last time: physically, power laws must be cut off at some limits. If, for example, we decide that the integral will really extend from $b_{\min}$ to $b_{\max}$, we get

$$\frac{dW}{d\omega dV dt} = \frac{16e^6}{3c^3 m^2 v} n_e n_i Z^2 \ln\left(b_{\max}/b_{\min}\right) \ . \tag{13}$$

You can do the usual checks: $e = 0$ implies no radiation, large m implies less acceleration and hence less radiation, and so on. You are in the presence of the famed *Coulomb logarithm* much spoken of in legend. This occurs a lot when Coulomb forces are involved, since the force scales as $1/r^2$ and the area element scales as r. The first thing to note is that logarithms are blessedly insensitive to precise values; $\ln(10^{16})$ is only twice $\ln(10^8)$, for example. That means that we don't have to be too precise in defining $b_{\rm max}$ and $b_{\rm min}$ to get a decent answer. But what is it that produces upper and lower cutoffs? Well, our approximation is invalid if $b \gg v/\omega$, so we can try $b_{\rm max} = v/\omega$. What about the other limit? In our derivation we assumed that the electron was basically going in a straight line path. This will clearly be invalid when $\Delta v \sim v$, which occurs when $b = b_{\rm min} = 4Ze^2/(\pi m v^2)$. Or, we could adopt a quantum approach: the classical calculation we've done is invalid when $b < h/mv$, since that would imply $\Delta x \Delta p < \hbar$, where $\Delta x \sim b$ and $\Delta p \sim mv$. Thus, we could also try $b_{\rm min} = h/mv$. Again, the insensitivity of the logarithm means we have latitude here. We will, of course, take the *larger* of the two possible values of $b_{\rm min}$.

One reason we have to use these approximations is that we're trying to do a classical calculation of a quantum process. We can sort of adjust for that by writing the exact results in terms of a *Gaunt factor* $g_{ff}(v,\omega)$:

$$\frac{dW}{d\omega dV dt} = \frac{16\pi e^6}{3\sqrt{3}c^3 m^2 v} n_e n_i Z^2 g_{ff}(v,\omega) \ , \tag{14}$$

where

$$g_{ff}(v,\omega) = (\sqrt{3}/\pi)\ln(b_{\rm max}/b_{\rm min}) \ . \tag{15}$$

The Gaunt factor is normalized in this way so that typically its value is near unity. Note, by the way, that the expression has a $1/v$ in it. **Ask class:** does that mean that for a given frequency ω the power per volume diverges for electrons of arbitrarily low velocity? No! We have to realize that the energy of the photons comes from the kinetic energy of the electrons, so if $\frac{1}{2}mv^2 < h\nu$ then a photon of energy $h\nu$ can't be created. There is thus another cutoff if you want the power at a particular frequency: only electrons in a certain velocity range contribute. This is called a photon discreteness effect.

Now that we have an answer for a particular velocity, we can integrate over an electron velocity distribution to get bulk emission from a region. Consider a thermal distribution, meaning that electron velocities are apportioned according to a Maxwell-Boltzmann distribution. Rybicki and Lightman get the integrated emission for an electron temperature T, in erg s^{-1} cm^{-3} Hz^{-1}:

$$\epsilon_\nu^{ff} = \frac{dW}{dV dt d\nu} = 6.8\times 10^{-38} Z^2 n_e n_i T^{-1/2} e^{-h\nu/kT} \bar{g}_{ff} \ , \tag{16}$$

where $\bar{g}_{ff}$ is the velocity-averaged Gaunt factor at temperature T.

Note that this is a rather "flat" spectrum for $h\nu < kT$, and that it rolls over exponentially for higher energies. That high-energy cutoff comes from photon discreteness and the lack of too many high-energy electrons. **Ask class:** given this spectrum, what is the rough frequency range where most of the energy will come out? Somewhere in the kT range. Note that if the electron distribution is nonthermal, one has to do different integrals.

If we integrate this emission over all frequencies, we get a total emission

$$\frac{dW}{dV\,dt} = 1.4 \times 10^{-27} T^{1/2} n_e n_i Z^2 \bar{g}_B \tag{17}$$

(in cgs units), where $\bar{g}_B$ is the frequency and velocity averaged Gaunt factor, usually in the range 1.1 to 1.5.

Now, we've discussed this in terms of an emission process. But there must therefore be a corresponding absorption process; in this case, free-free emission. We can get a free-free absorption coefficient from Kirchoff's law, and we find that

$$\alpha_\nu^{ff} = 3.7 \times 10^8 \mathrm{cm}^{-1} T^{-1/2} Z^2 n_e n_i \nu^{-3} \left(1 - e^{-h\nu/kT}\right) \bar{g}_{ff} \ . \tag{18}$$

Ask class: why is there a $1 - e^{-h\nu/kT}$ factor in there? It corrects for stimulated emission. The Rosseland mean of the free-free absorption coefficient is

$$\alpha_R^{ff} = 1.7 \times 10^{-25} T^{-7/2} Z^2 n_e n_i \bar{g}_R \ , \tag{19}$$

where quantities are measured in cgs units, and $\bar{g}_R$ is weighted as in the Rosseland mean and is of order unity. The particular form $T^{-7/2}$ for the Rosseland mean, or ν^{-3} for the frequency dependence, is called a *Kramers opacity* and occurs for bound-free as well as free-free.

The preceding was all done for nonrelativistic electrons. In certain very high-energy situations, though, the electron speeds can be relativistic. What is to be done? A cool way of approaching this problem is through the "method of virtual quanta". First, transform into a frame in which the electron is stationary and the ion is moving. The moving ion produces a "pulse" of electric field, which Compton scatters off of the electron. The radiation that emerges is the bremsstrahlung radiation as seen in this new frame. You can then Lorentz transform back into the "lab" frame where the electron is moving to get the radiation rate. Stuff like this is common in quantum electrodynamics. One can often find symmetries between apparently unrelated processes. For example, consider Compton scattering. A photon hits an electron, and bounces off; this means that if one plots the interaction on a time axis, then a photon and electron converge, then diverge. It happens, though, that a positron can be treated as an electron moving backward in time (that's Feynman's insight). This means that if you look at Compton scattering "sideways", two photons converge and produce an electron-positron pair, or the pair annihilates to produce two photons. Thus, there is an essential relation between Compton scattering and pair annihilation/production.

Enough of this diversion. When one puts in the relativistic effects, a reasonable fitting formula in cgs units is

$$\frac{dW}{dV\,dt} = 1.4\times10^{-27}T^{1/2}Z^2 n_e n_i \bar{g}_B(1+4.4\times10^{-10}T)\ . \tag{20}$$

One final set of comments. We have now discussed in some more detail two processes: Compton scattering and bremsstrahlung. As observers, we are often presented with a spectrum and we'd like to know the fundamental process that created it. In principle it may not sound too bad: look at the characteristics of the spectrum, then just identify it! In reality, though, if only continuum processes are operating (these vary only slowly with frequency), it can be difficult to do unique identification. Is your spectrum due to Comptonization, or is it the sum of many blackbodies of different temperatures? Be careful in these circumstances. Many, many people have a tendency to use the following approach: (1) assume some form or mechanism (e.g., bremsstrahlung), (2) derive parameters from the spectral fit (e.g., temperature or density), (3) assign great meaning to those derived parameters. The fact is that the derived parameters can vary significantly given different fits. A good fit does not guarantee physical meaning! If one has lines or edges it's usually easier, but beware of continuum fits.

Recommended Rybicki and Lightman problem: 5.1