Plasma Processes

Initial questions: We see all objects through a medium, which could be interplanetary, interstellar, or intergalactic. How does this medium affect photons? What information can we obtain?

In the preceding few lectures, we've focused on specific microphysical processes. In doing so, we have ignored the effect of other matter. In fact, we've implicitly or explicitly assumed propagation through a vacuum for most applications. It's time to take on matter!

When we introduced Maxwell's equations, we very solemnly defined $\mathbf{D} = \epsilon\mathbf{E}$ and $\mathbf{H} = \mathbf{B}/\mu$ to include the effects of matter, where ϵ is the dielectric constant and μ is the magnetic permeability. Is this necessary? No! Remember that Maxwell's equations explicitly include sources, in the form of ρ and $\mathbf{j}$. If we do this consistently, for all charges and currents (whether or not they are in a medium), then Maxwell's equations for $\mathbf{E}$ and $\mathbf{B}$ alone work just fine. Thus, Maxwell's equations for a "vacuum" work fine in that case, as long as both free and bound charges are included.

In that case, we can again think about the propagation of radiation, this time more generally. Again let's assume a space and time variation of the form $\exp i(\mathbf{k}\cdot\mathbf{r} - \omega t)$. Then we get

$$\begin{aligned} i\mathbf{k}\cdot\mathbf{E} &= 4\pi\rho\ , & i\mathbf{k}\cdot\mathbf{B} &= 0\ , \\ i\mathbf{k}\times\mathbf{E} &= i(\omega/c)\mathbf{B}\ , & i\mathbf{k}\times\mathbf{B} &= (4\pi/c)\mathbf{j} - i(\omega/c)\mathbf{E}\ . \end{aligned} \tag{1}$$

Remember, by the way, that our justification for using a $e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}$ variation is that Maxwell's equations are *linear* (there are no terms of the form E^2 or EB, for example). Thus every Fourier mode propagates independently. There are other physical theories (e.g., strong-field general relativity) that are *not* linear, which means that these modes would mix and could thus not be considered independently in this way. The linearity of Maxwell's equations is also why we can get away with using complex numbers; the real and imaginary parts never mix, so they can be considered to yield independent solutions based on the original equations.

Now, in general this could be pretty tough, if the medium is something arbitrary (air, water, glass). In our case, though, we're specifically interested in a *plasma*, which can be loosely defined as an ionized gas that is globally neutral. That means that all charges are mobile in principle. However, as we've done before, we'll assume that the ions are basically stationary for our purposes, and mainly serve to keep the plasma neutral. Another important simplifying assumption is that the plasma is isotropic. **Ask class:** what does that imply about the magnetic field? It implies that there is no significant external magnetic field, because that would break isotropy.

Let's consider nonrelativistic electrons. A given electron follows Newton's law

$$m\dot{\mathbf{v}} = -e\mathbf{E}\ . \tag{2}$$

There can be an internal magnetic field, just not an ordered one. **Ask class:** why, then, have we neglected the magnetic component of the force? It's because that term is of order v/c, which is small if the motion is nonrelativistic. Given our assumption about the space and time variations of quantities, this means

$$\mathbf{v} = e\mathbf{E}/(i\omega m)\ . \tag{3}$$

The current density is $\mathbf{j} = -ne\mathbf{v}$, meaning that we get

$$\mathbf{j} = \sigma\mathbf{E}\ , \tag{4}$$

where the conductivity is $\sigma = ine^2/(\omega m)$. This is Ohm's law; the current responds directly to the electric field. Note, however, that this statement requires isotropy. **Ask class:** can they think of a specific example in which anisotropy will lead to a different response to an applied electric field? If there is a strong magnetic field, charges and currents move along the field a lot better than across, so no longer is there this linear relation. In general, in fact, the conductivity is a tensor. However, for our situation we can treat it as a scalar due to isotropy.

From charge conservation, our $\exp i(\mathbf{k}\cdot\mathbf{r} - \omega t)$ assumption gives

$$-i\omega\rho + i\mathbf{k}\cdot\mathbf{j} = 0 \tag{5}$$

so that

$$\rho = \omega^{-1}\mathbf{k}\cdot\mathbf{j} = \sigma\omega^{-1}\mathbf{k}\cdot\mathbf{E}\ . \tag{6}$$

If we define the dielectric constant by

$$\epsilon \equiv 1 - 4\pi\sigma/(i\omega) \tag{7}$$

(note that this is real, since σ has an i in it), we get

$$\begin{aligned} i\mathbf{k}\cdot\epsilon\mathbf{E} &= 0\ , & i\mathbf{k}\cdot\mathbf{B} &= 0\ , \\ i\mathbf{k}\times\mathbf{E} &= i(\omega/c)\mathbf{B}\ , & i\mathbf{k}\times\mathbf{B} &= -i(\omega/c)\epsilon\mathbf{E}\ . \end{aligned} \tag{8}$$

Well, lookee here. This looks just like the "source-free" vacuum equations we had before, except for ϵ. Indeed, arguing as before, we find that $\mathbf{k}$, $\mathbf{E}$, and $\mathbf{B}$ are mutually perpendicular. However, we find that

$$c^2k^2 = \epsilon\omega^2\ . \tag{9}$$

Since ϵ depends on ω, we no longer have the simple vacuum situation in which all frequencies travel at the same rate, the phase velocity is the same as the group velocity, and so on.

Thus, the presence of a plasma introduces *dispersion*, where wave packets spread and there is effectively an index of refraction. If we substitute in expressions, we can rewrite the dielectric constant as

$$\epsilon = 1 - (\omega_p/\omega)^2 \; , \tag{10}$$

where $\omega_p = \sqrt{4\pi n e^2/m}$ is called the *plasma frequency.* Numerically, $\omega_p = 5.63 \times 10^4 n^{1/2}$ s^{-1} if n is in cm^{-3}.

Ask class: from these definitions and the dispersion relation, what does this tell us about propagation when $\omega < \omega_p$? It means that k is imaginary: $k = (i/c)\sqrt{\omega_p^2 - \omega^2}$. **Ask class:** what does that mean about the propagation of radiation below ω_p? It means that there is an exponential cutoff in the amplitude, with a distance scale of order $2\pi c/\omega_p$. Therefore, effectively, frequencies below ω_p can't propagate in a plasma.

Side note 1: one way to get quick intuition in a number of astrophysical situations is to have a number of characteristic quantities in mind. The plasma frequency is an example: if you have a plasma of a given number density and are considering propagation of electromagnetic waves, you should compare the frequency of the wave with the plasma frequency. If a magnetic field is involved, think of the cyclotron frequency. If a high density plasma is relevant, think of the Fermi energy. Stuff like that. It helps you decide quickly what regime you're in and what processes are likely to be relevant.

Side note 2: since σ is purely imaginary, Ohm's law means that there is a 90° phase shift between $\mathbf{j}$ and $\mathbf{E}$. Therefore, in a time-averaged sense, $\mathbf{j} \cdot \mathbf{E} = 0$ and there is no net work done by the field in an isotropic plasma. That also means there is no dissipation, so below the plasma frequency you have a pure reflection. Thus, you can probe the ionosphere of the Earth by finding out when a wave of a given frequency is completely reflected. You can also communicate intercontinentally by bouncing low-frequency waves off of the ionosphere.

Now back to our regular program. Electromagnetic radiation travels at a velocity different from c, due to the presence of matter. The phase velocity, $v_{\rm ph} \equiv \omega/k$, is greater than the speed of light. However, the physically relevant speed is the group velocity $v_{\rm gr} = c\sqrt{1 - \omega_p^2/\omega^2}$, which is less than c (this is the speed at which wave energy or information travel). The frequency dependence means that there is dispersion in the propagation of light over a variety of frequencies. One especially useful application is to pulsars. Suppose a pulsar is a distance d away. **Ask class:** how long does it take for light of a given frequency to reach us? The time is

$$t_p(\omega) = \int_0^d ds/v_g(\omega) \; . \tag{11}$$

Here s measures the line of sight distance to us. Plasma frequencies in the ISM are really low, usually 10^3 Hz or so, so we can assume $\omega \gg \omega_p$ and therefore

$$v_g^{-1} \approx \frac{1}{c}\left(1 + \frac{1}{2}\frac{\omega_p^2}{\omega^2}\right) \ . \tag{12}$$

Therefore, the propagation time is

$$t_p \approx \frac{d}{c} + (2c\omega^2)^{-1} \int_0^d \omega_p^2 \, ds \ . \tag{13}$$

This is the vacuum time (d/c) plus some extra. There is therefore a gradient in the time as a function of frequency,

$$dt_p/d\omega = -\frac{4\pi e^2}{cm\omega^3}\mathcal{D} \ , \tag{14}$$

where $\mathcal{D} \equiv \int_0^d n \, ds$ is the dispersion measure. In principle this can be used to find the distance to a pulsar, given an estimate of the average number density of plasma in the ISM. In practice, the errors are pretty large, because the interstellar medium has a lot of small-scale structure. This is especially true in directions that have a lot of plasma, such as towards the Galactic center.

Let's complicate things a bit by assuming we have a fixed external magnetic field $\mathbf{B}_0$ in the plasma. **Ask class:** qualitatively, what does that do? It means that the plasma and propagation in it are no longer isotropic, since the magnetic field introduces a preferred direction. It also means that not all polarizations are equal in their propagation properties.

We can make some progress by thinking about the special case of propagation along the direction of the field, and by considering only cold plasma (i.e., nonrelativistic motion). Also, let's assume that the magnitude of the external field is a lot greater than the magnitude of the fields of the propagating wave, so that the equation of motion becomes

$$md\mathbf{v}/dt = -e\mathbf{E} - \frac{e}{c}\mathbf{v} \times \mathbf{B}_0 \ . \tag{15}$$

For propagation along a fixed direction we only have to consider two polarization modes. Let's choose circular modes:

$$\mathbf{E}(t) = Ee^{-i\omega t}(\epsilon_1 \mp i\epsilon_2) \ , \tag{16}$$

where - gives right circular and + gives left circular polarization. If the wave propagates along $\mathbf{B}_0 = B_0\epsilon_3$, then by substituting we find that the steady-state velocity has the form

$$\mathbf{v}(t) = \frac{-ie}{m(\omega \pm \omega_B)}\mathbf{E}(t) \ , \tag{17}$$

where $\omega_B = eB_0/mc$ is the cyclotron frequency for nonrelativistic particles. To see this explicitly, let's assume that $\mathbf{B}_0 = B_0\epsilon_3$ (so that $\mathbf{B} \times \mathbf{E} = 0$; note that ϵ_1 etc. are unit vectors). Let us also make the guess that $\mathbf{v}(t) = C_1\mathbf{E}(t)$, where C_1 is a constant. Our equation then becomes

$$\begin{aligned}
C_1(-im\omega)Ee^{-i\omega t}(\epsilon_1 \mp i\epsilon_2) &= -eEe^{-i\omega t}(\epsilon_1 \mp i\epsilon_2) - (e/c)C_1Ee^{-i\omega t}\left[(\epsilon_1 \mp i\epsilon_2)\times\epsilon_3 B_0\right] \\
C_1(-im\omega)(\epsilon_1 \mp i\epsilon_2) &= -e(\epsilon_1 \mp i\epsilon_2) - (e/c)C_1(-\epsilon_2 \mp i\epsilon_1)B_0 \\
(-iC_1 m\omega)\epsilon_1 \mp (C_1 m\omega)\epsilon_2 &= [-e \pm iC_1(eB_0/c)]\epsilon_1 + [\pm ie + (eB_0/c)C_1]\epsilon_2\ .
\end{aligned} \tag{18}$$

Thus we have two equations, one for the ϵ_1 direction and one for the ϵ_2 direction:

$$\begin{aligned}
\epsilon_1 : -e + iC_1 m\omega \pm iC_1(eB_0/c) &= 0\ \Rightarrow -e + C_1[im\omega \pm i(eB_0/c)] = 0 \\
\epsilon_2 : \pm ie \pm C_1 m\omega + (eB_0/c)C_1 &= 0\ \Rightarrow \pm ie + C_1(eB_0/c \pm m\omega) = 0\ .
\end{aligned} \tag{19}$$

Both equations have the solution $C_1 = -ie/[m(\omega \pm eB_0/mc)]$, which gives us our previous expression.

Ask class: what are the implications of this, as the wave goes through the medium? Since the speeds of the different polarizations are different, there will be a net rotation of the polarization vector as the wave propagates through the plasma. One can rewrite this as an expression for the dielectric constant:

$$\epsilon_{R,L} = 1 - \frac{\omega_p^2}{\omega(\omega \pm \omega_B)}\ . \tag{20}$$

In general, an electric field vector with wavenumber $\mathbf{k}$ traveling a distance $\mathbf{d}$ will rotate in phase by $\mathbf{k}\cdot\mathbf{d}$. If the wavenumber is not constant along the path, this must be integrated.

Before getting the formula for Faraday rotation (the angle of rotation after some distance), let's think about some limits. **Ask class:** what do they expect the rotation to be if the field strength is very small? It should also be small. **Ask class:** does the equation satisfy this constraint? Yes, because small B_0 means small ω_B, and hence a small difference between the polarizations. **Ask class:** from the formula, what happens at extremely large B_0? There, also, the difference is small, because the ω_B in the denominator means that $\epsilon \to 1$ when B_0 is big.

As can be verified easily (see book), in the common astrophysical limits that $\omega \gg \omega_p$ and $\omega \gg \omega_B$ we have for our angle of rotation

$$\Delta\theta = \frac{2\pi e^3}{m^2c^2\omega^2}\int_0^d nB_\parallel\, ds\ . \tag{21}$$

Here $B_\parallel$ is the component of B parallel to the line of sight.

Now let's think about ways in which this might be applied in practice. First, **Ask class:** if the magnetic field is uniform (no change in direction or magnitude in the region), what happens to the *degree* of polarization due to Faraday rotation? Nothing; the angle simply

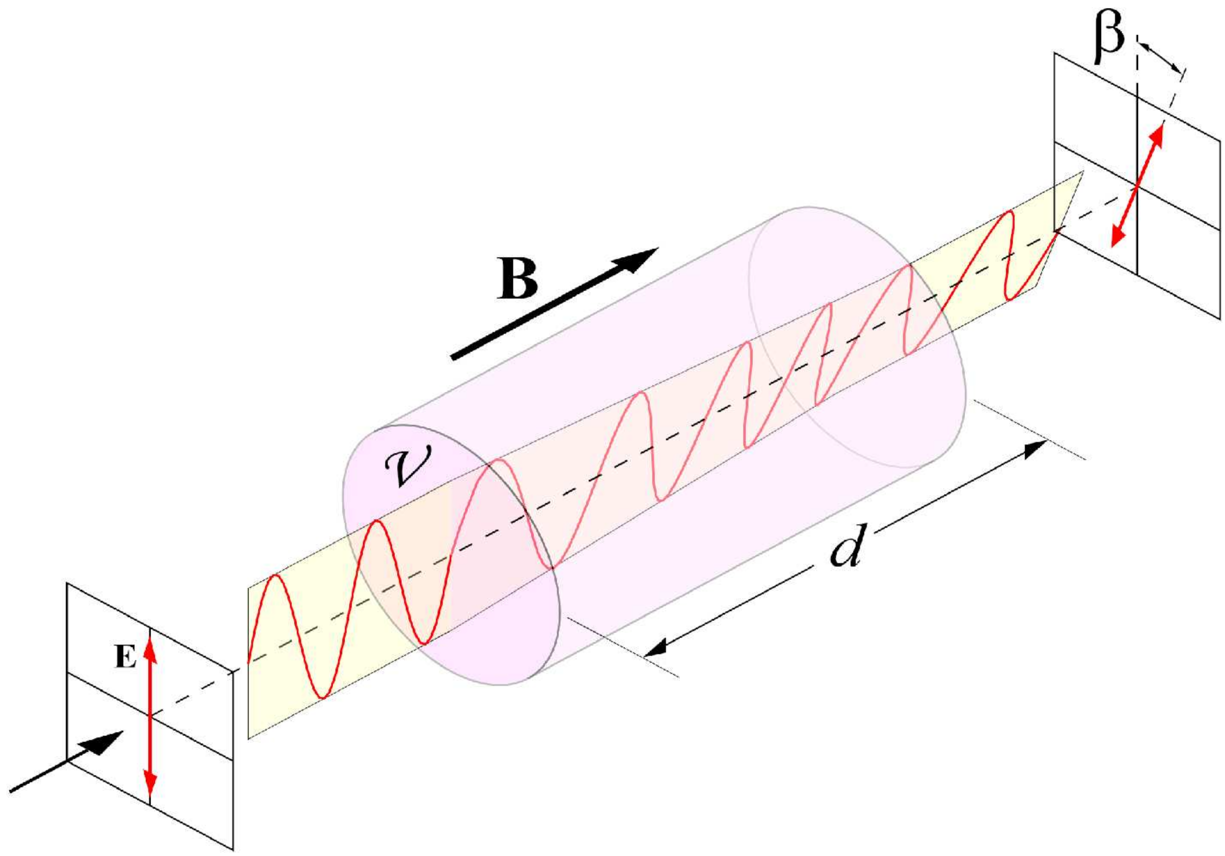

Fig. 1.— Schematic of Faraday rotation, from Wikipedia. This is for the rotation of polarization in matter, where there is a quantity ν related to the properties of that matter. In our case, the angle β of rotation is proportional to the integral of the number density times the magnetic field strength along the propagation direction over the extent of the region.

rotates. Okay, then, **Ask class:** what if the region has a tangled magnetic field and you observe a region that involves many tangles? Then, different parts are rotated by different amounts, so the net result is a decrease in the degree of polarization. This is sometimes used in observations of active galactic nuclei. Many times parts of the spectrum are thought to be due to synchrotron radiation, which you remember produces highly polarized light. However, observed with low angular resolution, there is little polarization. Observations of higher angular resolution do give net polarization, so by comparing the two one can estimate the line of sight integrated magnetic field. Similar methods are used to estimate the magnetic field strength in the interstellar medium or molecular clouds, when more direct spectroscopic information is unavailable.

Finally, we can just mention in passing one other effect of plasmas. Since the speed of electromagnetic waves is less than the vacuum speed of light, it becomes possible for particles to travel faster than the local speed of electromagnetic waves. This produces effects similar to shocks in the atmosphere, and generates Cerenkov radiation, which is bluish (i.e., it has a spectrum tipped towards high frequency, and would actually look blue to the eye). Only particles traveling faster than c/n emit radiation, which has been used to detect neutrinos (if they scatter electrons in water, the electrons can move faster than c/n) and estimate the energies of cosmic rays by using materials with different indices of refraction.

Recommended Rybicki and Lightman problem: 8.3