Orthonormal Tetrads, continued

Here's another example, that combines local frame calculations with more global analysis. Suppose you have a particle at rest at infinity, and you drop it radially into a Schwarzschild black hole. What is the radial velocity as seen by a local static observer at radius r? The particle being at rest at infinity means that its total energy is $u_t = -1$. Radial motion has θ and ϕ components zero, so $u^2 = -1$ means

$$\begin{aligned} u^r u_r + u^t u_t &= -1 \\ g_{rr}(u^r)^2 + g^{tt} u_t^2 &= -1 \\ (u^r)^2/(1-2M/r) - 1/(1-2M/r) &= -1 \\ (u^r)^2 &= 2M/r \ . \end{aligned} \tag{1}$$

Therefore, $u^r = dr/d\tau = 2M/r$. The radial velocity seen by a local static observer is

$$\begin{aligned} v^{\hat{r}} &= u^{\hat{r}}/u^{\hat{t}} \\ &= -u^{\hat{r}}/u_{\hat{t}} \\ &= -e^{\hat{r}}{}_r u^r/(e^t{}_{\hat{t}} u_t) \\ &= -(1-2M/r)^{-1/2} u^r/[(1-2M/r)^{-1/2} u_t] \\ &= \sqrt{2M/r} \ . \end{aligned} \tag{2}$$

Therefore, the locally measured radial velocity is just the same as the Newtonian expression, when Schwarzschild coordinates are used. By comparison, the radial velocity as measured at infinity is

$$v^r = \frac{dr}{dt} = u^r/u^t = u^r/(g^{tt}u_t) = (1-2M/r)u^r = (1-2M/r)\sqrt{2M/r} \ . \tag{3}$$

This drops to zero at the horizon. Note that there is one factor of $(1-2M/r)^{1/2}$ from the redshift and one from the change in the radial coordinate.

Geodesic Deviation and Spacetime Curvature

Previously we talked about geodesics, the paths of freely falling particles. We also indicated early on that the only "force" that gravity can exert on a particle is a tidal force. This is also another way to characterize the curvature of spacetime. In flat spacetime, two particles infinitesimally close to each other that are initially moving freely along neighboring paths will not deviate from each other. Similarly, two lines that are initially parallel in a plane remain parallel indefinitely. However, we know that tidal effects separate two particles in a gravitational field, in the same way that in a two-dimensional surface with curvature (e.g., a sphere or a hyperboloid), the separation between two initially parallel paths will change. It is this observation that initiated the study of non-Euclidean geometry in the early 1800s. We can therefore use geodesic deviation to study curvature.

Here we introduce one more bit of terminology: $D/d\lambda = \nabla_{\mathbf{u}}$, that is, $D/d\lambda$ is the directional derivative along the four-velocity. In component notation, for example, $DA^\alpha/d\lambda = A^\alpha{}_{;\mu} u^\mu$. Then, consider two nearby geodesics with separation vector $\mathbf{n}$. The "equation of geodesic deviation", which encodes tidal effects in general relativity, is (in component notation)

$$\frac{D^2 n^\alpha}{d\lambda^2} + R^\alpha{}_{\beta\gamma\delta} u^\beta n^\gamma u^\delta = 0 \ . \tag{4}$$

This equation defines the "Riemann curvature tensor", which can also be written as

$$R^\alpha{}_{\beta\gamma\delta} = \Gamma^\alpha{}_{\beta\delta,\gamma} - \Gamma^\alpha{}_{\beta\gamma,\delta} + \Gamma^\alpha{}_{\mu\gamma}\Gamma^\mu{}_{\beta\delta} - \Gamma^\alpha{}_{\mu\delta}\Gamma^\mu{}_{\beta\gamma} \ . \tag{5}$$

Here the Γs are the connection coefficients, as defined before. **Ask class:** what are the components of R in flat spacetime? Since $\Gamma^\alpha{}_{\beta\gamma} = 0$ in flat spacetime, $R^\alpha{}_{\beta\gamma\delta} = 0$.

From the Riemann curvature tensor one can form other tensors by contraction. For example, the Ricci curvature tensor is $R_{\mu\nu} = R^\alpha{}_{\mu\alpha\nu}$ and the scalar curvature is $R = R^\mu{}_\mu$. Incidentally, the scalar curvature measures the *local* presence of matter. That is, in a vacuum, even if that vacuum is just outside a star, $R = 0$. There is another contraction of the Riemann tensor (the Weyl tensor) that encodes "background" curvature.

The most important second-rank tensor one can form from the Riemann tensor is the Einstein curvature tensor

$$G^{\mu\nu} = R^{\mu\nu} - \frac{1}{2}\delta^{\mu\nu} R \ , \tag{6}$$

where δ is the Kronecker delta. The Einstein curvature tensor has deep geometrical significance; in particular, it satisfies the "contracted Bianchi identities"

$$G^{\mu\nu}{}_{;\nu} = 0 \ . \tag{7}$$

Stated in coordinate-free form, since this is the divergence of G, $\nabla \cdot \mathbf{G} = 0$, it says that G is a divergence-free tensor. It is also symmetric ($G^{\mu\nu} = G^{\nu\mu}$). These two properties are linked to the utility of G in the Einstein field equation, which links the curvature of space to the presence of matter. To understand this better we have to find a way to express the amount of matter and stresses in tensorial form.

The Stress-Energy Tensor

So far we have concentrated on the motion of test particles in curved spacetime. But "matter tells space how to curve", so we need a machine to quantify that as well. We'll examine this by defining the stress-energy tensor and looking at its components, then doing a couple of examples of what the stress-energy tensor is in a particular circumstance.

That machine is the stress-energy tensor, sometimes called the energy-momentum tensor. It is a symmetric second-rank tensor written $\mathbf{T}$, or in component form $T^{\alpha\beta}$. At a given location, the meaning of the components is as follows. Consider an observer with four-velocity u^α. That observer will see a density of four-momentum (i.e., four-momentum per unit of three-dimensional volume), of

$$dp^\alpha/dV = -T^\alpha_{\ \beta} u^\beta \ . \tag{8}$$

This can also be thought of as inserting the four-velocity into one of the slots of the stress-energy tensor: $\mathbf{T}(\mathbf{u}, \ldots) = \mathbf{T}(\ldots, \mathbf{u})$. That means that the n^α component of the four-momentum density is $\mathbf{n} \cdot d\mathbf{p}/dV = -T_{\alpha\beta} n^\alpha u^\beta$. Inserting the four-velocity into both slots gives the density of mass-energy measured in that Lorentz frame:

$$\mathbf{T}(\mathbf{u}, \mathbf{u}) = T_{\alpha\beta} u^\alpha u^\beta \ . \tag{9}$$

Finally, suppose we pick a particular Lorentz frame and choose two spacelike basis vectors $\mathbf{e}_j$ and $\mathbf{e}_k$. Then $\mathbf{T}(\mathbf{e}_j, \mathbf{e}_k) = T_{jk}$ is the j, k component of the stress as measured in that Lorentz frame. That is, T_{jk} is the j-component of force per unit area acting across a surface with a normal in the k direction, from $x^k - \epsilon$ to $x^k + \epsilon$. Symmetrically, it is also the k-component of force per unit area acting across a surface with a normal in the j direction, from $x^j - \epsilon$ to $x^j + \epsilon$. This means that the diagonal components ($j = k$) are the components of the pressure as measured in that Lorentz frame, and the off-diagonal components are the shear stresses.

Suppose we pick a particular observer with a particular Lorentz frame. Then what do the components mean? Here we'll use the notation (fairly widespread) that the "0" component is the time component.

$T^{00} = -T_0^{\ 0} = T_{00}$ is the density of mass-energy measured in that frame. $T^{j0} = T^{0j}$ is the volume density of the j-component of momentum, measured in that frame. Alternatively (and equivalently), T^{0k} is the k-component of the energy flux. Finally, T^{jk} is as defined before, which can also be thought of as the k-component of flux of the j-component of momentum.

Symmetry of the Stress-Energy Tensor

The stress-energy tensor must be symmetric. For a clever partial proof of this (involving the space components only), we can use an argument also used in Newtonian theory. Think of a very small cube, with a mass-energy density T^{00} and dimension L. The moment of inertia of the cube is $I \sim ML^2 \sim T^{00}L^5$. The torque on the cube from various stresses is simply the sum of the forces times the lever arms. For example, the z component of the torque is

$$N^z = (-T^{yz})L^2(L/2) + (T^{yx}L^2)(-L/2) - (-T^{xy}L^2)(L/2) - (T^{xy}L^2)(-L/2) \ . \tag{10}$$

That is, "the torque is the y-component of force on the $+x$ face times the lever arm to the $+x$ face, plus the y-component of the force on the $-x$ face times the lever arm to the $-x$

face, minus the x-component of the force on the $+y$ face times the lever arm to the $+y$ face minus the x-component of the force on the $-y$ face times the lever arm to the $-y$ face". Summed, this gives

$$N^z = (T^{xy} - T^{yx})L^3 \ . \tag{11}$$

Ask class: what would this (plus the moment of inertia) imply for the angular velocity induced as $L \longrightarrow 0$? The ratio $N/I \longrightarrow \infty$, so an infinitesimal cube would be set spinning infinitely fast unless $T^{xy} = T^{yx}$. Therefore, the spatial components are symmetric. So are the space-time components.

Example 1: Perfect Fluid

Let's think about a perfect fluid. This is defined as a fluid that has no shear stresses (e.g., no viscosity). In the "rest frame" of the fluid, where fluid motions are isotropic, there are therefore no off-diagonal components of the stress-energy tensor. Calling the mass-energy density in this frame ρ, we know that $T^{00} = \rho$. In addition, since fluid motions are isotropic in this frame, the pressure is the same for all three spacelike diagonal components; call it $T^{xx} = T^{yy} = T^{zz} = p$. This gives the stress-energy tensor in that special frame, but what about in general? The key trick here is to write a general tensorial equation that is valid in the special frame for which the calculation is easy. In this case, let the four-velocity of the fluid be u_α. In the rest frame of the fluid, recall that u_0 is the negative of the specific energy, which is just $u_0 = -1$. The rest of the four-velocity components vanish in this special frame. In addition, since this is a Lorentz frame, the spacetime metric is Minkowski, $g_{\alpha\beta} = \eta_{\alpha\beta}$. That means that in this special frame the equation

$$T_{\alpha\beta} = \rho u_\alpha u_\beta + p(\eta_{\alpha\beta} + u_\alpha u_\beta) \tag{12}$$

holds. Indeed, it still holds if we replace η with g, since $g = \eta$ in this frame. But then we have

$$T_{\alpha\beta} = p g_{\alpha\beta} + (p + \rho)u_\alpha u_\beta \ . \tag{13}$$

This is the general tensor equation we've been looking for, and it is valid in all frames! One can also write this in the form

$$\mathbf{T} = p\mathbf{g} + (p + \rho)\mathbf{u} \otimes \mathbf{u} \ . \tag{14}$$

This is a neat trick, but it isn't really all that mysterious. From experience in special relativity you know that if you have a quantity in one frame you can transform it easily enough. It's the same thing here: just remember that the metric tensor is Minkowski in the local Lorentz frame, and you're all set!

Test of understanding: suppose that an observer with a different four-velocity w^α measures the mass-energy density of this fluid. What is the result? Simply $T_{\alpha\beta}w^\alpha w^\beta$.

Example 2: Swarm of Particles

Suppose there is a group of noninteracting particles all moving with the same speed β in the x-direction. In their rest frame their mass density is ρ_0. What is the stress-energy as measured in the lab frame, where they move with velocity β?

We can use the same trick here to start. First, get the stress-energy tensor in a special frame. Then write it in such a way that it is a generally valid tensorial expression. In this case, in the rest frame there is no pressure and no shear stress, just a group of static particles with mass density ρ_0. Then $T^{00} = \rho_0$ and the rest of the components vanish. This means one could write $T^{\alpha\beta} = \rho u^\alpha u^\beta$, or $\mathbf{T} = \rho_0 \mathbf{u} \otimes \mathbf{u}$. Therefore, we also have $T_{\alpha\beta} = \rho_0 u_\alpha u_\beta$. This is valid in general.

The particular problem posed asks for the stress-energy as seen in a different frame. Since this frame is moving with constant velocity compared to the original Lorentz frame, the four-velocity of the new frame is found by a special relativistic transformation: $w^0 = \gamma u^0 = \gamma$, $w^x = -\gamma\beta u^0 = -\gamma\beta$, and $w^y = w^z = 0$. The stress-energy measured in the new frame is just $T^{\alpha\beta} = \rho_0 w^\alpha w^\beta$. This gives three nonzero components: $T^{00} = \rho_0 w^0 w^0 = \rho_0\gamma^2$; $T^{0x} = T^{x0} = -\rho_0\gamma^2\beta$; and $T^{xx} = \rho_0\gamma^2\beta^2$.

Conservation of Stress-Energy

Conservation laws are another of the unifying principles of physics. At various times mass, energy, mass-energy, momentum, angular momentum, and so on have been added to the list of conserved quantities. In general relativity all these conservation laws (and more besides!) are expressed in the general principle that the divergence of the stress-energy tensor vanishes: $\nabla \cdot \mathbf{T} = 0$ in coordinate-free notation. In flat space the components are $T^{\alpha\beta}{}_{,\beta} = 0$ (since $\mathbf{T}$ is symmetric, one could also take the divergence on the other index). In general spacetime, as usual the comma is replaced by a semicolon, so $T^{\alpha\beta}{}_{;\beta} = 0$. The meaning of this equation may also be expressed as "there are no sources or sinks of stress-energy", which is the same thing as saying that it is conserved.

Let's look at one example application of this equation, to see just how much information is bound up in it. Consider a nearly Newtonian perfect fluid, in which velocities are all much less than c and in its rest frame its pressure is tiny compared to its density. Then from before we have

$$\begin{gathered} T^{00} = (p+\rho)u^0u^0 - p \approx \rho \\ T^{0j} = T^{j0} = (\rho+p)u^0u^j \approx \rho v^j \\ T^{jk} = (\rho+p)u^ju^k + p\delta^{jk} \approx \rho v^j v^k + p\delta^{jk} \; . \end{gathered} \tag{15}$$

The time component of $\nabla \cdot \mathbf{T} = 0$ is (since spacetime is almost flat)

$$T^{00}{}_{,0} + T^{0j}{}_{,j} = 0 \;\Rightarrow\; \partial\rho/\partial t + (\rho v^j)_{,j} = 0 \; , \tag{16}$$

or $\partial\rho/\partial t + \nabla \cdot (\rho \mathbf{v}) = 0$, which is simply the equation of continuity! The spatial component is

$$T^{j0}{}_{,0} + T^{jk}{}_{,k} = 0 \;\Rightarrow\; \partial(\rho v^j)/\partial t + \partial(\rho v^j v^k)/\partial x^k + \partial p/\partial x^j = 0\ . \tag{17}$$

Using the equation of continuity, this becomes

$$\partial \mathbf{v}/\partial t + (\mathbf{v} \cdot \nabla)\mathbf{v} = -(1/\rho)\nabla p\ . \tag{18}$$

This is Euler's equation. Therefore, the equations for perfect fluids are one consequence of $\nabla \cdot \mathbf{T} = 0$.

The Einstein Field Equation

Now we're ready to put it all together, since we know how to quantify the presence of mass and energy and know how to characterized the curvature of spacetime. The final link is the relation between stress-energy and curvature, and this is provided by the *Einstein field equation*:

$$\mathbf{G} = 8\pi\mathbf{T} \tag{19}$$

where **G** is the Einstein tensor defined earlier. The proportionality constant is fixed by demanding that the theory agree with the Newtonian limits. This is the basis of all work in GR; effectively, by using things like the Schwarzschild spacetime we're using prior work that has solved the EFE for spherically symmetric vacuum spacetime. Incidentally, since **G** is symmetric, it has ten potentially unequal components that are determined by **T**. You might think this would be enough to uniquely determine all ten components of the (similarly symmetric, second-rank) metric tensor. This can't be, though, since there must be at least four degrees of freedom (since the local coordinates can be chosen arbitrarily). The resolution is that the Bianchi identities $G^{\alpha\beta}{}_{;\beta}$ mean that only six components are independent, so there are indeed four degrees of freedom.

When it was discovered that Einstein's theory predicted a dynamic universe (in contradiction to philosophy at the time and to the best observations), Einstein heavy-heartedly introduced a modification that he called the cosmological constant. When he introduced it, it took the form

$$\mathbf{G} + \Lambda\mathbf{g} = 8\pi\mathbf{T} \tag{20}$$

where **g** is the metric tensor. Current cosmologists prefer to think of it as a vacuum contribution to the stress-energy: $\mathbf{G} = 8\pi(\mathbf{T} + \mathbf{T}^{\rm VAC})$, where $\mathbf{T}^{\rm VAC} = -(\Lambda/8\pi)\mathbf{g}$. Einstein greatly regretted including this, since the universe is expanding, and called it the greatest blunder of his life. However, it looks more and more like this really may be the way the world operates! No one has a clue how this cosmological constant arises, though, and its value is a good 120 orders of magnitude lower than the simplest particle physics estimates would suggest.