Black Holes

We now embark on the study of black holes. Black holes are interesting for many reasons: they are one of only three possible endpoints of stellar evolution (the others being white dwarfs and neutron stars), they are the powerhouses of the most luminous things in the universe (quasars and active galactic nuclei), and they are the simplest macroscopic objects in the universe, with only two parameters important for their astrophysical properties. They are also way cool. Their simplicity means that it is possible to study them in a way impossible for any other object: with mathematical rigor. There was, for example, a flurry of activity in the late 1960s and early 1970s about proving theorems related to black holes, something which is mightily difficult to do with a star, for example! However, our main interest is in astrophysics, and specifically in explaining observed phenomena. We will therefore describe and use some of the derived results, but will not derive them (this would take overwhelmingly too much time). People with a desire to see the mathematical details can consult "The Mathematical Theory of Black Holes" by Chandrasekhar, or "Black Holes" by Novikov and Frolov, both of which are in our library.

Let us start by defining "black hole". A black hole is an object with an event horizon instead of a material surface. Events inside that horizon cannot be seen by any external observer. This is the fundamental property of black holes that distinguishes them from all other objects. It should be noted that (as we'll get to later) although there is compelling evidence for the existence of black holes in the universe, never has the existence of the horizon itself been demonstrated. An observation that unambiguously indicates the presence of a horizon would be a major advance. From time to time there are press releases announcing proofs of event horizons based on theoretical arguments, but so far these are unconvincing.

Inevitability of Collapse

One astrophysically relevant result to be stated is that once a star has compacted within a certain radius, formation of a black hole is inevitable. A basic reason for this is that in general relativity, all forms of energy gravitate. This includes pressure in particular. In a normal star, the pressure makes a tiny contribution to the total mass-energy, but in a very compact star the pressure is substantial. Normally, hydrostatic balance is produced by the offset of gravity by a pressure gradient, but in this case squeezing the star only increases the gravity (by increasing the pressure), so in it goes. The minimum stable radius for a spherically symmetric star is not the Schwarzschild radius $R_s = 2M$ as you might expect, but is $\frac{9}{8}R_s$.

Nonsingularity of R_s

When looking at the Schwarzschild geometry in Schwarzschild coordinates, one has the line element

$$ds^2 = -(1 - 2M/r)dt^2 + dr^2/(1 - 2M/r) + r^2(d\theta^2 + \sin^2\theta d\phi^2) \ . \tag{1}$$

This sure looks pathological at $r = 2M$. But you have to be careful. Perhaps it isn't the spacetime, but the coordinates that are at fault. For example, if you think about a sphere in normal (r, θ, ϕ) spherical coordinates, you might think that the North pole ($\theta = 0$) is a real problem, because the $d\phi^2$ coefficient goes to zero. But we know that this is just the coordinates; on a sphere, nothing at all is special about $\theta = 0$, as you can see by just redefining where your North pole is!

Now, $r = 2M$ is a special place; it's the location of the event horizon. But are things really singular there? In particular, is the curvature of the spacetime there finite or infinite, and would a freely falling observer feel finite or infinite tidal forces? **Ask class**: without actually computing the curvature, what is the right machinery in GR to use? The way to compute this is to define a local orthonormal frame and compute the components of the Riemann tensor (which, remember, tell you everything you need to know about the curvature). Then, boost into the freely falling observer's frame and figure out the tidal acceleration there. The net result is that all the components of tidal stress are $\sim M/r^3$, which is perfectly finite at $r = 2M$. In fact, the acceleration $\sim M^{-2}$ at the horizon, meaning that for a large enough black hole you could fall in without realizing it! You'd still be doomed, though. In contrast to this coordinate singularity at $r = 2M$, there is a real singularity at $r = 0$. There the tidal stresses are infinite, and anything that falls in gets munched regardless.

Ask class: What happens to the coordinates as you fall in to $r < 2M$? Looking at the line element with $r < 2M$, you see that the *sign* of the dr^2 term becomes negative, and the sign of the dt^2 term becomes positive. This means (as it turns out) that inside the event horizon the radius becomes a timelike coordinate, and the time becomes a spacelike coordinate. Specifically, that means that once inside $2M$, you *must* go to smaller radii, just as now you *must* go forward in time. You can't even move a centimeter outwards once you're inside, and avoiding the singularity at $r = 0$ is just as impossible as avoiding Monday.

This is a major difference between the modern conception of black holes and the pre-GR ideas sometimes linked to it. In 1783 John Michell realized that a star with 250 times the radius of the Sun that had an average density equal to that of the Earth would be dark according to Newton's theory. That's because the escape velocity would be the speed of light, so he imagined light climbing up, slowing down, and falling back. He would, however, have thought it possible to escape from such a star in a rocket. Not so in the modern conception. **Ask class**: for fun, how would we compute the radius of an object of mass M with an escape velocity equal to the speed of light, in the Newtonian limit? The escape velocity is $v^2 = 2GM/r$, so $v^2 = c^2$ means $r = 2GM/c^2$, just the same as the Schwarzschild radius!

No Hair Theorem

So far we've spent a lot of time with the Schwarzschild geometry, due to its simplicity. But how relevant is it, really? **Ask class**: thinking about Newtonian gravity, what are some factors other than the total mass that could influence the gravitational field outside a normal star? Quadrupole terms, fluid motions, asymmetries, et cetera. What happens when collapse into a black hole occurs? An amazing set of theorems proved in the early 1970's shows that the final result is a black hole that has only three qualities to it at all. These are mass, angular momentum, and electric charge. Everything else (quadrupole terms, magnetic moments, weak forces, etc.) decays away. This is a remarkable result that simplifies treatment of black holes greatly. One heuristic way to think about this relates to what you would see if you dropped a lightbulb into the black hole. As the lightbulb fell, light from it diminishes more and more in apparent intensity. **Ask class**: suppose we have a lightbulb with rest-frame specific intensity I_0. How do we compute the specific intensity seen at infinity when the bulb is at radius r, if the bulb falls radially from rest at infinity? The key here is to remember the $I \propto \nu^4$ law; tracking the frequency will allow computation of the specific intensity. There are two components to the frequency shift. One is the ordinary Doppler shift as seen by a local static observer, the other is the gravitational redshift from there to infinity. One must then trace the rays to get the final intensity.

Very soon, nothing more is left; in fact, the luminosity seen by a distant observer goes like

$$L \propto \exp\left(-\frac{t}{3\sqrt{3}M}\right) \; . \tag{2}$$

For a solar mass black hole the time constant is a few tens of microseconds. Therefore, in the blink of an eye the black hole really does appear black. In a somewhat analogous fashion, other properties of the infalling matter, such as magnetic field and lumpiness of the matter distribution, decay away on a similar timescale. Only mass, angular momentum, and charge are left.

It was discovered in 1963 that an exact spacetime exists for a black hole with just mass and angular momentum (Kerr geometry), and in 1965 a solution including charge was found (Kerr-Newman geometry). The most common coordinates used to express this spacetime are generalizations of Schwarzschild coordinates called Boyer-Lindquist coordinates, and for the record the metric line element is then

$$ds^2 = -(\Delta/\rho^2)[dt - a\sin^2\theta\, d\phi]^2 + (\sin^2\theta/\rho^2)[(r^2+a^2)d\phi - a\, dt]^2 + (\rho^2/\Delta)dr^2 + \rho^2\, d\theta^2 \; . \tag{3}$$

There are several definitions here. The parameter $a = J/M$ describes the angular momentum, and it has dimensions of mass. $\Delta = r^2 - 2Mr + a^2 + Q^2$, where Q is the electric charge (in cgs units Q^2 has the units of erg-cm, which can then be converted to grams in the usual geometrized units way). Finally, $\rho^2 = r^2 + a^2 \cos^2\theta$.

The most important new feature of this geometry, compared to Schwarzschild, is the $d\phi dt$ terms. These indicate a relation between time and azimuthal angle, and correspond to frame-dragging: spacetime is "twisted" in the direction of rotation of the black hole.

This geometry has a horizon (and therefore descibes a black hole) only if $Q^2 + a^2 \leq M^2$. If equality holds, this is called an extremal black hole. If this condition is violated, centrifugal acceleration or electrostatic repulsion will halt the collapse. You cannot, however, spin up a black hole or feed charge to it so that it loses its horizon.

Let's see if we even need the charge term, astrophysically. **Ask class**: how should we determine whether Q can ever be gravitationally significant? Suppose that $Q^2 = M^2$, the maximum possible. Converting M^2 into erg-cm units means $Q^2 = (Mc^2)(GM/c^2) = GM^2$. Suppose we compare the electrical and gravitational forces on a particle of mass m and charge q at a distance $r \gg M$, so the Newtonian force law is accurate. The electrical force is qQ/r and the gravitational force is GMm/r, so the ratio is $f_e/f_g = qQ/(GMm) = qG^{1/2}M/(GMm) = q/(G^{1/2}m)$. For example, for a proton $f_e/f_g \approx 10^{18}$ and for an electron $f_e/f_g \approx 2 \times 10^{21}$. This shows that (as always!) the electromagnetic force is overwhelmingly stronger than gravity if there is a lot of unbalanced charge. The result is that if Q is anything remotely significant gravitationally, the black hole will sweep up every stray charge within parsecs until it is almost electrically neutral. That's why we can ignore the charge, and consider just the mass and angular momentum when thinking about the spacetime. With angular momentum but no charge this is called the Kerr spacetime. It is also common to use the dimensionless quantity $j = a/M$ instead of a.

Properties of Kerr Spacetime

The Kerr spacetime is a lot more complicated than the Schwarzschild spacetime, but because it can be written in closed form it is still simple enough for fairly rigorous mathematical analysis. First of all, though, let's see what quantities are conserved in the Kerr spacetime. **Ask class**: will the squared four-velocity change? No, this is completely general, so we always have $u^2 = -1$ for massive particles, $u^2 = 0$ for photons. What else, though? **Ask class**: is the geometry spherically symmetric? No, because there is a preferred axis (the axis of rotation). Looking at the metric coefficients, **Ask class** what variables never appear? t and ϕ, so this is a stationary and axisymmetric spacetime. **Ask class** make a guess about which two conserved quantities result from these symmetries. Energy and angular momentum, respectively. There is also a fourth quantity, called Carter's fourth constant of motion, that is conserved, but it is more complicated than is worth writing down.

As said before, the major new component to this spacetime compared to Schwarzschild is *frame-dragging*, more as one gets closer to the hole. One of many bizarre consequences is that if you were to drop a particle from infinity radially at the hole, then as it got closer it would acquire a nonzero angular velocity (but still have zero angular momentum!). The angular velocity of a zero angular momentum particle, which can be thought of as the angular velocity of spacetime, is

$$\omega = \frac{2Mar}{(r^2 + a^2)^2 - a^2 \Delta \sin^2 \theta} \ . \tag{4}$$

For almost all applications of interest, the r^4 term dwarfs the others and $\omega \approx 2Ma/r^3 = 2jM^2/r^3$.

Frame-dragging has many implications. One is that, near enough to the hole, a particle *must* rotate in the same direction as the hole. This is even true outside the horizon, so there is a region, called the *ergosphere*, in which no static observers can exist; nonetheless, they could escape from that region, so it isn't like the horizon. The radius of the ergosphere is $r_{\rm ergo} = M + (M^2 - a^2 \cos^2 \theta)^{1/2}$. In addition, the black hole itself shrinks; the radius of the horizon is $r = M + \sqrt{M^2 - a^2}$, so for an extremally rotating black hole ($a = M$), $r = M$. The radius of the innermost stable circular orbit shrinks for prograde orbits (to a minimum of $r_{\rm ISCO} = M$ for $a = M$) and increases for retrograde orbits (to a maximum of $r_{\rm ISCO} = 9\,M$ for $a = M$). That means that if gas spirals in to the hole on prograde orbits, the energy emitted and hence the accretion efficiency increases with increasing spin (from 5.7% for $a = 0$ to 42% for $a = M$, or 40% if you discount energy that goes down the hole). Yet another consequence is that a particle in a circular orbit that is tilted with respect to the spin plane will precess in its orbit, at the rate ω. This means that a nonaxisymmetric warp in an accretion disk has a tough time surviving unless it is confined to a small radial range, because the strong dependence of ω on r means that there would be a lot of shear otherwise. Also, a gyroscope with an axis tilted from the spin axis will precess at ω. This is an effect which Gravity Probe B will try to detect, and that some people think has already been seen from some neutron star and black hole sources (although I'm highly skeptical). Finally, Kepler's Third Law (angular velocity of a particle in a circular orbit at r) takes the simple form

$$\Omega = \pm \frac{M^{1/2}}{r^{3/2} \pm a M^{1/2}} \tag{5}$$

where $+$ is for prograde and $-$ is for retrograde orbits.

Black Hole Thermodynamics

There is a remarkable black hole analogy with thermodynamics. If one computes the area of the horizon, it is

$$A = 8\pi M \left[M + (M^2 - a^2)^{1/2} \right] \ . \tag{6}$$

For Schwarzchild, $a = 0$, the area is $A = 16\pi M^2$ as expected. Hawking proved that in any interaction of a black hole or between black holes, the sum of the areas can never decrease. This leads one to a possible computation of the maximum amount of energy that can be radiated in a collision between black holes. For example, if two Schwarzschild black holes of mass M hit head-on, then you know that $16\pi M_{\rm tot}^2 \geq 32\pi M^2$, so $M_{\rm tot} > M/\sqrt{2}$ and no more than 29% of the total mass-energy can be radiated away. The best case would be two extremal Kerr black holes of the same mass and opposite angular momentum, for which the theoretical maximum is 50%. However, the *actual* amount radiated is much less than this, and must be computed numerically. For head-on Schwarzschild the efficiency is more like 0.1%.

The area theorem is awfully reminiscent of the second law of thermodynamics. But this would require that black holes have finite temperature, so that they radiate. When Bekenstein suggested the thermodynamic analogy, most people (including Hawking) were dubious, but then Hawking showed that black holes *do* radiate! This happens because virtual pairs of particles and antiparticles can be made real by the tidal acceleration near the event horizon, and on occasion one escapes while the other is sucked in; the effect is that the black hole "radiates" even though nothing escapes from inside the event horizon. This is an astrophysically unimportant effect because the effective temperature is $T \approx 10^{-7}\,{\rm K}(M/M_\odot)$, so a solar-mass black hole lasts about 10^{67} years. We'll never see a black hole radiate unless tiny ones (mass of a mountain) were formed in the early universe. Nonetheless, Hawking radiation does have importance in other ways. For example, a couple of years ago great excitement was produced when it was shown that the rate and spectrum of Hawking radiation from special black holes (Schwarzschild and extremally rotating) could be reproduced in M-theory, which is the best current candidate for the theory of everything. Hawking radiation also brings up interesting semi-philosophical questions; for example, particles and antiparticles have an equal likelihood of being emitted, whereas the star that formed the black hole and almost anything that fell in it were formed of particles. Thus, lepton and baryon number conservation seem to be violated by Hawking radiation.