Energy Losses and Gravitational Radiation

Energy loss processes

Now, we need to consider energy loss processes. **Ask class:** what are ways in which high-energy particles can lose energy? Generically, can lose energy by interacting with photons or other particles, or by interacting with a field and radiating. Let's break it down. First, interaction with photons (inverse Compton scattering). **Ask class:** would they expect protons or electrons to lose more energy by scattering with photons? Electrons have a much larger cross section, so they do. Now, interaction with particles. Electrons can interact with other electrons or with protons, but at ultrarelativistic energies these cross sections are relatively small. For protons, as we saw in the last lecture they can scatter off of other protons or nuclei. In such interactions the strong force is involved, and the cross section is something like $\sim 10^{-26}$ cm^2. **Ask class:** what does the column depth have to be for optical depth unity? The reciprocal of this, or about 10^{26} cm^{-2}. That's a lot! So, in a dense environment, collisions with particles can sap energy from high-energy particles. But what if the environment is low-density? Then, as we've argued, acceleration of particles to ultrarelativistic energies means that magnetic fields are present. In fact, the fields need to be strong enough to confine the particles as well (otherwise they escape without further gain of energy). Therefore, interaction of the particles with the field can produce radiation, diminishing the particle energy.

Curvature radiation.—First, suppose that the particle is forced to move along magnetic field lines with curvature radius R. As we found in the last lecture, the power radiated by a particle moving at a Lorentz factor $\gamma \gg 1$ is $P \approx (2e^2c/3R^2)\gamma^4$. A relativistic particle of charge e moving along an electric field $\mathcal{E}$ receives a power $e\mathcal{E}c$ from the field. Equating the power gain and power loss, the limiting Lorentz factor is

$$\gamma < 2 \times 10^7 R_8^{1/2} \mathcal{E}_4^{1/4} \ , \tag{1}$$

where $R = 10^8 R_8$ cm and $\mathcal{E} = 10^4 \mathcal{E}_4$sV cm^{-1}. For a supermassive black hole with $R = 10^{14}$ cm, γ might get up to 2×10^{10} (note the weak dependence on $\mathcal{E}$; also, the required magnetic field for $\mathcal{E}_4 \gg 1$ would be unrealistically high for a supermassive black hole). This would give a proton an energy of 2×10^{19} eV, which sounds high but is a factor of 10 short of the highest observed energies. For a neutron star with $R = 10^6$ cm and $\mathcal{E} = 10^{12}$, $\gamma < 2 \times 10^8$. So it sounds bad. **Ask class:** but is curvature radiation always relevant? What if you have a very high-energy particle? Then, the particle doesn't follow the field lines. Remember that curvature radiation and synchrotron radiation are both just types of acceleration radiation. If a particle is moving in a straight line, it radiates very little. So, the question is whether synchrotron radiation keeps the particle along the field lines.

Synchrotron radiation.—From the last lecture, proton synchrotron radiation causes an energy decay at a rate $\dot{E}/E \approx 3 \times 10^5 \gamma B_{12}^2$ s^{-1} if the highly relativistic protons move perpendicular to field lines. If $E/\dot{E}$ is short compared to the time needed to travel across the acceleration region, the proton will follow field lines. Near a strongly magnetic neutron star, with $B = 10^{12}$ G, there's no chance of highly relativistic protons moving across field lines. Even for a supermassive black hole, where $B \sim 10^5$ G, a proton with $\gamma = 10^{11}$ as required will be forced to follow the field lines in a time much less than the crossing time of the region, so in both cases it is likely that protons will be forced to follow field lines.

This is a problem. My best guess is that there are AGN with field geometries such that the curvature radius is much greater (factor of >100!) than the distance from the hole. That would just barely allow the observed energies. If future detectors (such as the Pierre Auger Array) see cosmic rays of energies much greater than the current record holders, this would be a real problem. We'll get into other issues with the highest energy cosmic rays near the end of the course.

Gravitational radiation

As direct detection of gravitational radiation draws nearer, it is useful to consider what such detections will teach us about the universe. The first such detection, of course, will be of immediate significance because it will be a direct confirmation of a dramatic prediction of general relativity: to paraphrase John Wheeler, that spacetime tells sources how to move, and moving sources tell spacetime how to ripple.

Beyond this first detection, gravitational wave detections will pass into the realm of astronomy, allowing new observational windows onto some of the most dynamic phenomena in the universe. These include merging neutron stars and black holes, supernova explosions, and possibly echoes from the very early history of the universe as a whole. They are also anticipated to provide the cleanest tests of predictions of general relativity in the realm of strong gravity.

However, there are important differences from standard astronomy. In electromagnetic observations, in every waveband there are sources so strong that they can be detected without knowing anything about the source. You don't need to understand nuclear fusion in order to see the Sun! In contrast, as we will see, most of the expected sources of gravitational radiation are so weak that sophisticated statistical techniques are required to detect them at all. These techniques involve matching templates of expected waveforms against the observed data stream. Maximum sensitivity therefore requires a certain understanding of what the sources look like, hence of the characteristics of those sources. In addition, when detections occur, it will be important to put them into an astrophysical context so that the implications of the discoveries are evident.

Before discussing types of sources, though, we need to have some general perspective on how gravitational radiation is generated and how strong it is. We will begin by discussing radiation in a general context.

By definition, a radiation field must be able to carry energy to infinity. If the amplitude of the field a distance r from the source in the direction (θ, ϕ) is $A(r, \theta, \phi)$, the flux through a spherical surface at r is $F(r, \theta, \phi) \propto A^2(r, \theta, \phi)$. If for simplicity we assume that the radiation is spherically symmetric, $A(r, \theta, \phi) = A(r)$, this means that the luminosity at a distance r is $L(r) \propto A^2(r) 4\pi r^2$. Note, though, that when one expands the static field of a source in moments, the slowest-decreasing moment (the monopole) decreases like $A(r) \propto 1/r^2$, implying that $L(r) \propto 1/r^2$ and hence no energy is carried to infinity. This tells us two things, regardless of the nature of the radiation (e.g., electromagnetic or gravitational). First, radiation requires time variation of the source. Second, the amplitude must scale as $1/r$ far from the source.

We can now explore what types of variation will produce radiation. We'll start with electromagnetic radiation, and expand in moments. For a charge density $\rho_e(\mathbf{r})$, the monopole moment is $\int \rho_e(\mathbf{r}) d^3r$. This is simply the total charge Q, which cannot vary, hence there is no electromagnetic monopolar radiation. The next static moment is the dipole moment, $\int \rho_e(\mathbf{r}) \mathbf{r} d^3r$. There is no applicable conservation law, so electric dipole radiation is possible. One can also look at the variation of currents. The lowest order such variation (the "magnetic dipole") is $\int \rho_e(\mathbf{r}) \mathbf{r} \times \mathbf{v}(\mathbf{r}) d^3r$. Once again this can vary, so magnetic dipole radiation is possible. The lower order moments will typically dominate the field unless their variation is reduced or eliminated by some special symmetry.

Now consider gravitational radiation. Let the mass-energy density be $\rho(\mathbf{r})$. The monopole moment is $\int \rho(\mathbf{r}) d^3r$, which is simply the total mass-energy. This is constant, so there cannot be monopolar gravitational radiation. The static dipole moment is $\int \rho(\mathbf{r}) \mathbf{r} d^3r$. This, however, is just the center of mass-energy of the system. In the center of mass frame, therefore, this moment does not change, so there cannot be electric dipolar radiation in this frame (or any other, since the existence of radiation is frame-independent). The equivalent of the magnetic dipolar moment is $\int \rho(\mathbf{r}) \mathbf{r} \times \mathbf{v}(\mathbf{r}) d^3r$. This, however, is simply the total angular momentum of the system, so its conservation means that there is no magnetic dipolar gravitational radiation either. The next static moment is quadrupolar: $I_{ij} = \int \rho(\mathbf{r}) r_i r_j d^3r$. This is not conserved, therefore there can be quadrupolar gravitational radiation.

This allows us to draw general conclusions about the type of motion that can generate gravitational radiation. A spherically symmetric variation is only monopolar, hence it does not produce radiation. No matter how violent an explosion or a collapse (even into a black hole!), no gravitational radiation is emitted if spherical symmetry is maintained. In addition, a rotation that preserves axisymmetry (without contraction or expansion) does not generate gravitational radiation because the quadrupolar and higher moments are unaltered. Therefore, for example, a neutron star can rotate arbitrarily rapidly without emitting gravitational radiation as long as it maintains axisymmetry.

This immediately allows us to focus on the most promising types of sources for gravitational wave emission. The general categories are: binaries, continuous wave sources (e.g., rotating stars with nonaxisymmetric lumps), bursts (e.g., asymmetric collapses), and stochastic sources (i.e., individually unresolved sources with random phases; the most interesting of these would be a background of gravitational waves from the early universe).

We now make some order of magnitude estimates. What is the approximate expression for the dimensionless amplitude h of a metric perturbation, a distance r from a source? We argued that the lowest order radiation had to be quadrupolar, and hence depend on the quadrupole moment I. This moment is $I_{ij} = \int \rho r_i r_j d^3x$, so it has dimensions MR^2, where M is some mass and R is a characteristic dimension. We also argued that the amplitude is proportional to $1/r$, so we have

$$h \sim MR^2/r \ . \tag{2}$$

We know that h is dimensionless, so how do we determine what else goes in here? In GR we usually set $G = c = 1$, which means that mass, distance, and time all have the same effective "units", but we can't, for example, turn a distance squared into a distance. Our current expression has effective units of distance squared (or mass squared, or time squared). We note that time derivatives have to be involved, since a static system can't emit anything. Two time derivatives will cancel out the current units, so we now have

$$h \sim \frac{1}{r}\frac{\partial^2(MR^2)}{\partial t^2} \ . \tag{3}$$

Now what? To get back to physical units we have to restore factors of G and c. It is useful to remember certain conversions: for example, if M is a mass, GM/c^2 has units of distance, and GM/c^3 has units of time. Playing with this for a while gives finally

$$h \sim \frac{G}{c^4}\frac{1}{r}\frac{\partial^2(MR^2)}{\partial t^2} \ . \tag{4}$$

Since G is small and c is large, the prefactor is *tiny*! That tells us that unless M and R are large, the system is changing fast, and r is small, the metric perturbation is minuscule.

Let's make a very rough estimate for a circular binary. Suppose the total mass is $M = m_1 + m_2$, the reduced mass is $\mu = m_1 m_2/M$, the semimajor axis is a, and the orbital frequency Ω is therefore given by $\Omega^2 a^3 = M$. Without worrying about precise factors, we say that $\partial^2/\partial t^2 \sim \Omega^2$ and $MR^2 \sim \mu a^2$, so

$$h \sim (G/c^4)(1/r)(\mu M/a)\ . \tag{5}$$

This can also be written in terms of orbital periods, and with the correct factors put in we get, for example, for an equal mass system

$$h \approx 10^{-22} \left(\frac{M}{2.8\,M_\odot}\right)^{5/3} \left(\frac{0.01\ \mathrm{sec}}{P}\right)^{2/3} \left(\frac{100\ \mathrm{Mpc}}{r}\right)\ , \tag{6}$$

which is scaled to a double neutron star system. This is really, really, small: it corresponds to less than the radius of an atomic nucleus over a baseline the size of the Earth. That's why it is so challenging to detect these systems!

Gravitational Wave Detectors

This is an immense subject; multiple current detectors are in existence (although not yet at a sensitivity likely to detect astrophysical sources), and many things are currently being learned. We will therefore only discuss the very basics of laser interferometer detectors.

Since gravitational waves at lowest order are quadrupolar instead of dipolar, the polarization patterns are different from what we are used to for electromagnetic radiation. That is, you recall that if an EM wave is traveling in the z direction, then all polarizations are combinations of a linear polarization in the x direction, and a linear polarization in the y direction (rotation with time gives elliptical polarization in general). Gravitational waves are also transverse (i.e., there is no component in the propagation direction, here taken along the z axis), but the fundamental modes are what are known as the + and $\times$ modes.

To picture these, consider an initially circular ring of test particles in the x-y plane. As a + mode gravitational wave passes through along the z axis, the ring first compresses along the y axis and stretches along the x axis (conserving area to first order), then compresses along the x axis and stretches along the y axis, then repeats. The $\times$ mode is the same thing but rotated by 45°.

Detection thus requires sensitivity to changes in separations. However, an absolute measurement of the distance is hopeless: from above, a fractional change $h \sim 10^{-22}$ might be typical, so for a baseline of 4 km (such as in the LIGO detectors), the actual change in length is only 4×10^{-17} cm. A proton has a radius of about 10^{-13} cm! What can be done? The best answer appears to be to use laser interferometry. Imagine two interferometric cavities at right angles to each other. Laser light goes down both, hits mirrors, then comes back to interfere. Very slight changes in the relative distances along the cavities can be detected by movement of the interference fringes. The changes in distance can be measured to much less than a wavelength in the same way that the centroid location of a star can be measured to much less than the resolution of an instrument, by taking advantage of high laser power.

However, there are complications. High laser power also means that the shot noise of the photons causes fluctuations in the mirror. It can also cause thermal distortions of the mirrors, as well as (potentially) producing a type of parametric resonance in the cavity because mirror vibrations couple to the radiation pressure and can produce oscillations in this way. For space-based interferometers such as the planned LISA mission, the mirrors drift freely, meaning that their mutual distance changes constantly. The net result is that there are a scary number of issues to deal with, but at this point none of them seem fatal in the long run, and in fact ground-based instruments such as LIGO have reached their initial design sensitivity, so things are looking good!

Additional references: *Gravitation* by Misner, Thorne, and Wheeler is the classic reference for all things related to general relativity. Also, several years ago Caltech had a gravitational wave course and all the materials (including videos of the lectures) are online:

http://elmer.caltech.edu/ph237/