Tensor manipulations

One final thing to learn about tensor manipulation is that the metric tensor is what allows you to raise and lower indices. That is, for example, $v_\alpha = g_{\alpha\beta}v^\beta$, where again we use the summation convention. Similarly, $v^\alpha = g^{\alpha\beta}v_\beta$, where $g^{\alpha\beta}$ is the matrix inverse of $g_{\alpha\beta}$: $g^{\alpha\beta}g_{\beta\gamma} = \delta^\alpha_\gamma$, where δ is the Kronecker delta (1 if $\alpha = \gamma$, 0 otherwise).

Another special tensor is the Levi-Civita tensor $\epsilon_{\alpha\beta\gamma\delta}$. This tensor is defined as being completely antisymmetric. In flat spacetime, $\epsilon_{0123} = 1$ if 0 is the positive time direction and (123) is a right-handed set of spacetime basis vectors (e.g., xyz). Then $\epsilon_{\alpha\beta\gamma\delta} = 1$ if $(\alpha\beta\gamma\delta)$ an even permutation of 0123, -1 if it is an odd permutation, and 0 if any two indices are the same. In curved spacetime, we define the metric determinant $g = \det||g_{\alpha\beta}|| < 0$. Then, the Levi-Civita tensor is $\epsilon_{\alpha\beta\gamma\delta} = (-g)^{1/2}[\alpha\beta\gamma\delta]$ where $[\alpha\beta\gamma\delta]$ is $+1$ for an even permutation, -1 for an odd permutation, and zero if two indices are equal, as before. Note that the Levi-Civita tensor may be familiar from cross products: $\mathbf{A} \times \mathbf{B} = \epsilon_{ijk}A^iB^j$.

Proper time and four-velocity

As we move into discussion of particular spacetimes, we need to introduce two important concepts. The first is proper time. Consider a particle with nonzero rest mass, and suppose that we measure the invariant interval between to events on the world line of that particle. In general, for any coordinate system and observer we have $ds^2 = g_{\alpha\beta}dx^\alpha dx^\beta$. Remember that α and β run over all four spacetime indices, so that even though "x" reminds us of spatial coordinates, one of the indices indicates time. How is this perceived by an observer who is riding along with the particle? In that coordinate system, the particle is always at the spatial origin, meaning that all the spatial components of dx^α are zero. As a result, only the observer's time changes. We label this time as τ and call it the *proper time*. In this local frame the coordinate system is Minkowski, hence the metric coefficient is just -1 ($-c^2$ in physical units), therefore $ds^2 = -d\tau^2$. Note that because of our choice of metric signature $ds^2 < 0$ for a particle of nonzero rest mass, hence $d\tau^2 > 0$ as it should be.

Our second concept is the four-velocity u^α: $u^\alpha = dx^\alpha/d\tau$. Note, therefore, that one of the components of the four-velocity is $u^t = dt/d\tau$. This is the rate at which the *coordinate* time t passes relative to the *proper* time τ.

What about for a photon? This is trickier, since $ds^2 = d\tau^2 = 0$ for a photon. We therefore need another way to describe the motion of a photon. The way to do this is to simply define some parameter, call it λ, which checks off the location on a given path of motion (this is called an affine parameter, and must be chosen such that $dx^\alpha/d\lambda$ has constant magnitude along the path). We then look at the motion with respect to that parameter, meaning that $u^\alpha = dx^\alpha/d\lambda$. We will explore some consequences of this later.

Spacetime and metrics

Now let's get a little more concrete, which will eventually allow us to introduce additional concepts. Let's concentrate on one particularly important geometry, the Schwarzschild geometry (aka spacetime). It is very useful, not least because it is more general than you might think. It is the geometry outside of (i.e., in a vacuum) any spherically symmetric gravitating body. It is *not* restricted to static objects; for example, it is the right geometry outside a supernova, if that supernova is kind enough to be spherically symmetric. To understand some of its aspects we'll write down the line element (i.e., the metric in a particular set of coordinates). However, the coordinates themselves are tricky, so let's start with flat space.

Ask class: what is the Minkowski spacetime in spherical coordinates?

$$ds^2 = -dt^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \ . \tag{1}$$

Ask class: what is the meaning of each of the coordinates (this is not a trick question!)? Everything is as you expect: θ and ϕ are the usual spherical coordinates, r is radius, t is time. In particular, if you have two things at r_1 and r_2 (same t, θ, and ϕ) then the distance between them is $|r_2 - r_1|$. No sweat. You could also say that the area of a sphere at radius r is $4\pi r^2$.

Well, why all this rigamarole? It's because in Schwarzschild spacetime things get trickier. Now let's reexamine the Schwarzschild line element

$$ds^2 = -(1 - 2M/r)dt^2 + dr^2/(1 - 2M/r) + r^2(d\theta^2 + \sin^2\theta d\phi^2) \ . \tag{2}$$

Ask class: are the meanings of θ and ϕ changed? No, they're the same as always. This is guaranteed by the assumption of spherical symmetry that comes into the Schwarzschild derivation. But what about r? **Ask class:** suppose $dt = d\theta = d\phi = 0$. What is the proper distance between r_1 and r_2? In this case, $ds = g_{rr}^{1/2}dr$, so the distance is

$$D = \int_{r_1}^{r_2} g_{rr}^{1/2}dr = \int_{r_1}^{r_2}(1 - 2M/r)^{-1/2}dr \ . \tag{3}$$

That means that if $r_1 = 2M$, for example, the radial distance measured is rather different than in flat space! But if you calculate the area of a sphere of radius r, you get $4\pi r^2$ as usual, and the circumference of a circle is $2\pi r$ as usual. This is one of the most extreme geometric indicators of the curved spacetime: "π"=circumference/diameter drops like a rock!

What about time? **Ask class:** what is the relation between proper time $d\tau$ at r and the coordinate time dt if $dr = d\theta = d\phi = 0$? $d\tau^2 = -ds^2 = -g_{tt}dt^2 \Rightarrow d\tau = (1 - 2M/r)^{1/2}dt$. Therefore, as $r \to 2M$, the elapsed proper time is tiny compared to the elapsed coordinate time. It turns out that t, the coordinate time, is the time as seen at infinity. Therefore, to a distant observer it looks like an object falling into the horizon takes an infinite time to do so. This is the origin of the term "frozen star" used by many until the 1970's for black holes. You might think, then, that if you were to look at a black hole you'd see lots of frozen surprised aliens just outside the horizon. You actually would not, but more on that later. Another thing to think about is that when $r < 2M$, bizarre stuff happens, for example the signs of g_{tt} and g_{rr} switch. What does this signify? It could be just that the coordinate system is not useful there, the same way that at the north or south pole of a sphere the azimuthal coordinate ϕ changes arbitrarily rapidly for a finite linear motion. Or it could be that something truly horrible occurs at $r = 2M$, e.g., that any observer crossing inside is vaporized in a cloud of coordinates. The truth, as it happens, lies in between those extremes.

Conserved Quantities in Schwarzschild Spacetime

Let's take another look at the Schwarzschild spacetime. It is spherically symmetric. It is also stationary, meaning that nothing about the spacetime is time-dependent (for example, the time t does not appear explicitly in the line element). Now, in general in physics, any time you have a symmetry you have a conserved quantity. **Ask class:** for a particle or photon moving under just the influence of gravity, what are some quantities that will be conserved in the motion of that particle? As with any time-independent central force, energy and angular momentum will be conserved. These follow from, respectively, the symmetry with time and the symmetry with angle. In addition, the rest mass is conserved (more on that later). We have three conserved quantities, and four components to the motion, so if we knew one more we'd be set. Luckily, having spherical symmetry means that we can define a plane of motion for a single particle, so we only have three components to the motion (in particular, we might as well define the plane of motion to be the equatorial plane, so that $\theta = \pi/2$ and we don't have to worry about motion in the θ direction). That means we get a great boost in following (and checking) geodesic motion in the Schwarzschild spacetime from the conserved quantities.

Test particle.—Here we pause briefly to define the concept of a *test particle*. This is useful in thinking about the effect of spacetime on the motion of objects. A test particle is something that reacts to fields or spacetime or whatever, but does not affect them in turn. In practice this is an excellent approximation in GR whenever the objects of interest have much less mass than the mass of the system; this applies, for example, to gas in accretion disks.

We can now think some more about four-velocity. This is a good time to show a useful technique, which is the computation of quantities in a reference frame where they are particularly simple.

We'll start with the four-momentum p^α. From special relativity, you know that the time component is E, and the space component is the ordinary 3-momentum, call it $\mathbf{P}$. Let's consider the square of this four-momentum, $p^\alpha p_\alpha = g_{\alpha\beta} p^\alpha p^\beta$. Assume we have gone into a local Lorentz frame, so that $g_{\alpha\beta} = \eta_{\alpha\beta}$. Then $p^2 = p^\alpha p_\alpha = -E^2 + P^2$. Of course, we can boost into another Lorentz frame in which the particle is not moving (assuming it's not a photon). **Ask class:** what's the square of the four-momentum then? In that frame, $P = 0$, so the square is just $-E^2$. But then the total energy is just the rest mass energy, so $p^\alpha p_\alpha = -m^2c^4$. Now, the last step. **Ask class:** what kind of geometric object is $p^\alpha p_\alpha$? Is it a scalar, vector, tensor, what? No free indicies, so this is a scalar and its value is the same in any Lorentz frame. Therefore, in general $p^2 = -m^2$, so we end up with $-m^2 = -E^2 + P^2$, or $E^2 = m^2 + p^2$. With units, it's just the famous equation $E^2 = m^2c^4 + p^2c^2$.

If you divide the four-momentum by the mass m you get the four-velocity u, at least for particles with nonzero rest mass. This means that for massive particles, $u^2 = u^\alpha u_\alpha = -1$. This leads to a cool proof. **Ask class:** what is the derivative of $u^\alpha u_\alpha$? Since it's the derivative of -1, a constant, the derivative is zero! But if you write it out, the derivative is also twice the dot product of the four-velocity with the four-acceleration. Therefore, the four-acceleration is always orthogonal to the four-velocity (this is true in general, not just for motion under gravity).

What about a photon? There, the four-momentum is again $p^2 = -E^2 + P^2$, but since for a photon $E = Pc$, we always have $p^2 = 0$. Similarly, for a photon the squared four-velocity is $u^2 = u^\alpha u_\alpha = (dx^\alpha/d\lambda)(dx_\alpha/d\lambda) = g_{\alpha\beta}(dx^\alpha/d\lambda)(dx^\beta/d\lambda) = (g_{\alpha\beta}dx^\alpha dx^\beta)/(d\lambda)^2$. But the numerator is just the interval traveled by the photon. **Ask class:** what is that interval? It's zero, so (consistent with the squared four-momentum being zero) we find that the squared four-velocity of a photon is zero.

This, therefore, is true for *any* spacetime at all, and even in the presence of other arbitrary forces. For a particle with mass, the squared four-velocity is $u^2 = -1$, and for a photon or other massless particle $u^2 = 0$.

Let's see an example of how this can help us. In general, for a massive particle, $u^t = dt/d\tau$, $u^r = dr/d\tau$, $u^\theta = d\theta/d\tau$, and $u^\phi = d\phi/d\tau$. In Schwarzschild coordinates it is also true that $u_t = -e$, where e is the specific energy (energy per mass) of the particle ($e = 1$ for a particle at rest at infinity) and u_ϕ is the specific angular momentum of the particle. Suppose we have a particle in circular motion, although not necessarily Keplerian. Then there is no r or θ motion and $u^2 = -1$ gives us $u^t u_t + u^\phi u_\phi = -1$. We can put this into a more convenient form by writing $(g^{t\alpha}u_\alpha)u_t + (g^{\phi\alpha}u_\alpha)u_\phi = -1$. The Schwarzschild spacetime is diagonal, so this becomes simply $g^{tt}(u_t)^2 + g^{\phi\phi}(u_\phi)^2 = -1$. Consulting the line element, we find $g^{tt} = -1/(1 - 2M/r)$ and $g^{\phi\phi} = 1/r^2$, so the specific energy is

$$e = \sqrt{(1 - 2M/r)(1 + u_\phi^2/r^2)}\ . \tag{4}$$

For example, for a slowly rotating star with $u_\phi \approx 0$, the energy is $e = \sqrt{1 - 2M/r}$. This means that a particle of mass m originally at infinity will release a total energy $m(1-e)$ if it finally comes to rest on the star's surface. Let's check if this makes sense in the Newtonian limit $r \gg M$. Then $e \approx 1 - M/r$, so the energy released is mM/r, which is the Newtonian form.

Additional references: As usual, go to Misner, Thorne, and Wheeler, Gravitation, for more details.