Orthonormal Tetrads

Let's now return to a subject we've mentioned a few times: shifting to a locally Minkowski frame. In general, you want to take a metric that looks like $g_{\alpha\beta}$ and shift into a frame such that locally the metric is $\eta_{\alpha\beta} = (-1, 1, 1, 1)$. It is conventional to represent the new coordinates with hats (e.g., $\hat{t}$, $\hat{r}$, $\hat{\theta}$, $\hat{\phi}$), so that

$$ds^2 = -d\hat{t}^2 + d\hat{r}^2 + d\hat{\theta}^2 + d\hat{\phi}^2 \ . \tag{1}$$

The transformation from the local to the global coordinates is done with the transformation matrices $e^{\hat{\alpha}}{}_{\beta}$ and $e^{\alpha}{}_{\hat{\beta}}$. For example, $u^\alpha = e^{\alpha}{}_{\hat{\beta}} u^{\hat{\beta}}$. The components of the transformation matrices come from the transformation of the metric tensor:

$$\eta_{\hat{\alpha}\hat{\beta}} = e^{\mu}{}_{\hat{\alpha}} e^{\nu}{}_{\hat{\beta}} g_{\mu\nu} \ . \tag{2}$$

The inverse transformation matrices are computed similarly:

$$g_{\mu\nu} = e^{\hat{\alpha}}{}_{\mu} e^{\hat{\beta}}{}_{\nu} \eta_{\hat{\alpha}\hat{\beta}} \ . \tag{3}$$

This is especially easy for the Schwarzschild metric, because the metric is diagonal. Then, for example, $e^{t}{}_{\hat{t}} = 1/\sqrt{-g_{tt}} = (1 - 2M/r)^{-1/2}$ and $e^{\phi}{}_{\hat{\phi}} = 1/\sqrt{g_{\phi\phi}} = r^{-1}$. One can get the inverse matrices similarly, and the special diagonal nature of Schwarzschild means that, unsurprisingly, you get the reciprocals: $e^{\hat{t}}{}_{t} = \sqrt{-g_{tt}} = (1 - 2M/r)^{1/2}$ and $e^{\hat{\phi}}{}_{\phi} = \sqrt{g_{\phi\phi}} = r$ (here, as usual, we implicitly assume that we have chosen the orbital plane to be at $\theta = \pi/2$, which the spherical symmetry allows us to do).

Note that even after having transformed into a reference frame in which the spacetime is as Minkowski as possible (i.e., first but not second derivatives vanish), there is still freedom to choose the coordinates. Also, remember that there is always freedom to have Lorentz boosts; that is, having found a frame in which the spacetime looks flat, another frame moving at a constant velocity to the first also sees flat spacetime. This means that your choice of frame ("orthonormal tetrad") is based to some extent on convenience. Around a spherical star, a good frame is often the static frame, which is the frame that is unmoving with respect to infinity.

Something you may have noticed is that although the new coordinates are called $\hat{t}$, $\hat{r}$, $\hat{\theta}$, and $\hat{\phi}$, the last two are not actually angular coordinates. In fact, we have sneakily switched into a Cartesian system; in fact, this is an *orthonormal tetrad* because $g_{\alpha\beta} = \eta_{\alpha\beta}$. The "angular-looking" coordinates $\hat{\theta}$ and $\hat{\phi}$ are therefore normal linear coordinates that happen, at that particular location, to be along the unit vectors parallel to θ and ϕ.

With this in mind, it is important to note the following: *Velocities measured by local observers in this frame are not the four-velocities (e.g., $u^{\hat{r}} = d\hat{r}/d\tau$), but are instead the* *derivatives relative to the local time* $\hat{t}$*, e.g.,* $v^{\hat{r}} = d\hat{r}/d\hat{t} = u^{\hat{r}}/u^{\hat{t}}$. This is sometimes a point of confusion because it is often productive to think of the four-velocity u^α or $u^{\hat{\alpha}}$ as somehow a more fundamental quantity. However, on reflection it should make sense because the speed measured by any observer is computed by the distance that observer saw something move divided by the time the observer measured the movement to take. This is neither more nor less than dx/dt, where x and t are measured locally.

Now let's see some additional examples. Suppose a particle moves along a circular arc with a linear velocity in the ϕ direction $v^{\hat{\phi}}$ as seen by a static observer at Schwarzschild radius r. What is the angular velocity as seen at infinity? The Schwarzschild time t is the time at infinity, and the angle ϕ has a unique meaning everywhere, so the angular velocity as seen at infinity is just $\Omega = d\phi/dt$. In order to relate this to the locally measured quantities, we note that $v^{\hat{\phi}} = d\hat{\phi}/d\hat{t} = u^{\hat{\phi}}/u^{\hat{t}}$. But this is $e^{\hat{\phi}}{}_\phi u^\phi / \left[e^{\hat{t}}{}_t u^t\right]$. Since $\Omega = d\phi/dt = u^\phi/u^t$, then

$$\Omega = (e^{\hat{t}}{}_t / e^{\hat{\phi}}{}_\phi) v^{\hat{\phi}} = \left(\frac{v^{\hat{\phi}}}{r}\right)(1 - 2M/r)^{1/2} \ . \tag{4}$$

This makes sense; it's just the same as one would calculate in the Newtonian limit, except that the frequency is less because of redshifting.

Equation of Motion and Geodesics

So far we've talked about how to represent curved spacetime using a metric, and what quantities are conserved. Now let's see how test particles move in such a spacetime. To do this, we need to take a brief diversion to see how derivatives are changed in curved spacetime.

First, let's simplify and go back to thinking about a flat two-dimensional plane. Consider a scalar quantity. Define that quantity to be Φ. Let's think in terms of a Cartesian coordinate basis. The derivative along the x direction is $\partial\Phi/\partial x$, and along the y direction is $\partial\Phi/\partial y$. We will adopt the notation that a comma denotes a partial derivative, so these become $\Phi_{,x}$ and $\Phi_{,y}$, respectively. If we want the directional derivative along some vector $\mathbf{v}$ with components v^x and v^y, that derivative is $\partial_\mathbf{v}\Phi = \Phi_{,x}v^x + \Phi_{,y}v^y = \Phi_{,\alpha}v^\alpha$. Similarly, if we have a vector field $\mathbf{A}$, its gradient is the rank 2 tensor $\nabla A = A^\alpha_{,\beta}$ and its directional derivative along $\mathbf{v}$ is the rank-1 tensor $A^\alpha_{,\beta}v^\beta$. As always, the geometric objects formed (e.g., gradient or directional derivative) have coordinate-free reality, but we express them here in terms of components for concreteness. A particle moving in free space (no forces on it) simply continues moving along in a straight line. That means that the directional derivative of the velocity, along the velocity, is zero (this comes from, e.g., conservation of momentum and energy). Therefore, $u^\alpha_{,\beta}u^\beta = 0$ is the equation of motion for a free particle. All these results are also true for free motion in flat spacetime.

But what happens when the spacetime is curved? Then the coordinates themselves twist and turn. One must take this into account by adding another term to the equation for a derivative (you can think of this as a chain rule, if you like; the derivative of a quantity contains one term for the partial derivative of that quantity, and another for the partial derivative of the coordinates themselves). A result that we'll simply have to state is that the correction terms involve the *connection coefficients*. For a coordinate basis, these are

$$\Gamma^{\alpha}_{\beta\gamma} = \frac{1}{2} g^{\alpha\mu} \left(g_{\mu\beta,\gamma} + g_{\mu\gamma,\beta} - g_{\beta\gamma,\mu} \right) \ . \tag{5}$$

The actual derivative itself is represented by a semicolon. The connection coefficients come in with a + sign when the index to be "corrected" is up, and a − sign when the index is down. Some examples:

$$\begin{array}{ll} \Phi_{;\alpha} = \Phi_{,\alpha} & v^{\alpha}_{;\beta} = v^{\alpha}_{,\beta} + \Gamma^{\alpha}_{\mu\beta} v^{\mu} \\ v_{\alpha;\beta} = v_{\alpha,\beta} - \Gamma^{\mu}_{\alpha\beta} v_{\mu} & T^{\beta}_{\alpha;\gamma} = T^{\beta}_{\alpha,\gamma} + \Gamma^{\beta}_{\mu\gamma} T^{\mu}_{\alpha} - \Gamma^{\mu}_{\alpha\gamma} T^{\beta}_{\mu} \ . \end{array} \tag{6}$$

Note that for scalar fields there is no correction.

What about "straight lines" in curved spacetime? These are called geodesics, and are simply extensions of what we found in flat spacetime. That is, the directional derivative of the velocity along itself is zero, or $\nabla_{\mathbf{u}}\mathbf{u} = 0$, so $u^{\alpha}_{;\beta} u^{\beta} = 0$. If one defines some affine parameter λ along the geodesic (the proper time τ is a good choice for particles with nonzero rest mass), then this comes out to

$$\frac{d^2 x^{\alpha}}{d\lambda^2} + \Gamma^{\alpha}_{\mu\gamma} \frac{dx^{\mu}}{d\lambda} \frac{dx^{\gamma}}{d\lambda} = 0 \ . \tag{7}$$

From the definition of the four-velocity, $u^{\alpha} = dx^{\alpha}/d\lambda$, we can also write this

$$\frac{du^{\alpha}}{d\lambda} + \Gamma^{\alpha}_{\mu\gamma} u^{\mu} u^{\gamma} = 0 \ . \tag{8}$$

We'll explore some consequences of this in a second, but first a quick check that this is reasonable. This is the equation of motion. So, **Ask class:** what kind of path should it give in flat space? It should give a straight line, meaning that the acceleration $du^{\alpha}/d\lambda = 0$. Is it so? Let's look at the connection coefficients again. **Ask class:** what is the Minkowski metric? $\eta_{\alpha\beta} = (-1, 1, 1, 1)$. So, **Ask class:** what are the derivatives of the components of $\eta_{\alpha\beta}$? The derivatives are all zero. **Ask class:** what does that imply for the connection coefficients? They're related to derivatives of the metric coefficients, so the connection coefficients are zero. Therefore, **Ask class:** what is the equation of motion in flat spacetime? It becomes just $du^{\alpha}/d\lambda = 0$, a straight line, as required.

For a Schwarzschild spacetime, let's consider motion in the $r\phi$ plane (i.e., no θ motion, which one can always arrange in a spherically symmetric spacetime just by redefinition of coords). Then the radial equation of geodesic motion is (here we use the proper time τ as our affine parameter)

$$\frac{d^2r}{d\tau^2} + \frac{M}{r^2} - (1 - 3M/r)u_\phi^2/r^3 = 0 \; . \tag{9}$$

In deriving this we've used the fact that the specific angular momentum $u_\phi = g_{\phi\alpha}u^\alpha = g_{\phi\phi}u^\phi = r^2(d\phi/d\tau)$. Let's think about what all this means. First, let's check this in the Newtonian limit. In that limit, $M/r \ll 1$ and can be ignored, and at low velocities $d\tau^2 \approx dt^2$ so we get the usual expression

$$\frac{d^2r}{dt^2} = -\frac{M}{r^2} + u_\phi^2/r^3 \; . \tag{10}$$

In particular, that means that for a circular orbit, $d^2r/dt^2 = 0$, the specific angular momentum is given by $u_\phi^2 = Mr$.

What about in strong gravity? First consider radial motion, $u_\phi = 0$. Then $d^2r/d\tau^2 = -M/r^2$. This has the same form as the Newtonian expression, but remember that the coordinates mean different things, so you have to be careful. Now consider circular orbits. Again we set $d^2r/d\tau^2 = 0$, to find $u_\phi^2 = Mr^2/(r - 3M)$. But wait! Something's strange here. That $r - 3M$ in the denominator means that the specific angular momentum goes to infinity at $r = 3M$, but the horizon is at $r = 2M$. Have we reached a contradiction of some sort?

No, but we have happened upon one of the most important features and predictions of general relativity. As you will demonstrate in the problem set, all of this implies that the specific angular momentum has a minimum at a radius well outside the horizon. This is also the minimum radius of a circular orbit that is stable to small perturbations, hence is called the innermost stable circular orbit (ISCO).

Ask class: what does this mean for gas spiraling close to a black hole or neutron star? It means that even if the gas was moving in almost circular orbits at larger distances, then (neglecting other forces) when it reaches this critical radius it'll go right in without having to lose more angular momentum. This radius is called plays a fundamental role in the physics of accretion disks around very compact objects.

Qualitatively, one can think of it like this. A fundamental feature of the Schwarzschild geometry is the so-called "pit in the potential". That is, near a compact object gravity is "stronger" than you would have expected based on an extrapolation of the Newtonian law. To compensate for this, the angular velocity has to be higher than it would have been otherwise, so the angular momentum is higher than it would have been in the Newtonian limit, and eventually u_ϕ reaches a minimum and then increases as the radius is decreased further. This predicted behavior is an example of a phenomenon that only occurs in strong gravity, and so can only be tested by observing compact objects.

If we plug the u_ϕ for a circular orbit into the formula for specific energy we found earlier, we find

$$-u_t(\mathrm{circ}) = \frac{r-2M}{\sqrt{r(r-3M)}}\ . \tag{11}$$

At the ISCO, $-u_t = \sqrt{8/9}$, so 5.7% of the binding energy is released in the inspiral to this point.

Now let's do another sample calculation. We argued above that to compensate for the stronger gravity, particles have to move faster near a compact object. That would suggest that the angular velocity observed at infinity would be higher than in Newtonian gravity. However, there is also a redshift, which decreases frequencies. Let's calculate the frequency of a circular orbit observed at infinity, to see which effect wins.

We said a while back that the Schwarzschild time coordinate t is simply the time at infinity, and the azimuthal coordinate ϕ is also valid at infinity (in fact, unlike t, ϕ has constant meaning at all radii). Therefore, the angular velocity is $\Omega = d\phi/dt$. To calculate this, we relate it to components of the four-velocity: $d\phi/dt = (d\phi/d\tau)/(dt/d\tau) = u^\phi/u^t$. Now, we express this in terms of our conserved quantities u_ϕ and u_t:

$$\frac{u^\phi}{u^t} = \frac{g^{\alpha\phi}u_\alpha}{g^{\alpha t}u_\alpha} = \frac{g^{\phi\phi}u_\phi}{g^{tt}u_t} = \frac{u_\phi/r^2}{-u_t/(1-2M/r)}\ . \tag{12}$$

Then

$$\Omega = \frac{1-2M/r}{r^2}\frac{u_\phi}{-u_t} = \frac{1-2M/r}{r^2}\frac{\sqrt{Mr^2/(r-3M)}}{(r-2M)/\sqrt{r(r-3M)}} = \sqrt{M/r^3}\ . \tag{13}$$

This is exactly the Newtonian expression! By a lovely coincidence, in Schwarzschild coordinates the angular velocity observed at infinity is exactly the same as it is in Newtonian physics.

We've taken a long diversion here to discuss the radial component of the equation of geodesic motion and some of its implications. Let's briefly consider the azimuthal component, specifically $u_{\phi;\alpha}u^\alpha = 0$. This can be expressed as

$$\frac{du_\phi}{d\tau} + ??? = 0\ . \tag{14}$$

We can certainly go through the same procedure of calculating the connection coefficients. But here is a place where we should apply our intuition to shortcut those calculations. Recalling that u_ϕ is the specific angular momentum, and that we are considering geodesic motion (no nongravitational forces), **Ask class:** what should the "???" be in this equation for the Schwarzschild spacetime? It should be zero! Angular momentum is conserved for

Schwarzschild geodesics, so $du_\phi/d\tau$ had better vanish. You can confirm this explicitly if you want.

One last note about geodesics is that they represent extrema in the integrated path length ds^2 between two events. The reason for this is extremely deep and ultimately comes down to the same reason that optical paths are extrema in length. Basically, if you represent light as a wave, then two paths with different lengths will have different numbers of cycles and hence different phases along the way. With different phases, there is destructive interference and the amplitude is small. Only near an extremum, where nearby paths differ in path length by a small second-order quantity, are the phases close to each other, so only there is the interference constructive and the amplitude high. For massive particles the principle is the same, according to quantum mechanics. Again, a particle can be represented by a wave (or a wave function), and again if nearby paths have significantly different phases the interference will be destructive. Only near an extremum is the amplitude high. For this case, however, it isn't simply the length of the path, but instead the integral of the Lagrangian that matters (this integral is called the action). Extremization of the action is one of the unifying principles of physics, and provides (for example) a different way of looking at general relativity than the geometric approach we've taken.

We can write the four-acceleration a^α as $a^\alpha = u^\alpha_{;\beta}u^\beta$, so that the geodesic equation of motion is $a^\alpha = 0$. If there is a force f^α on a particle of rest mass m, the resulting equation of motion (and the fully general one, in general relativity) is $a^\alpha = f^\alpha/m$, which is a nice return to good old $F = ma$!

Additional references: You guessed it: go to Misner, Thorne, and Wheeler, Gravitation, for more details.