Accretion Disks: angular momentum

Now let's go back to black holes. Black holes being what they are, something that falls into one disappears without a peep. It might therefore seem that accretion onto a black hole would release no energy. It isn't the case, however, and the reason has to do with angular momentum.

Thinking in terms of Newtonian gravity, suppose you have a point source of mass. **Ask class:** taking gravity into effect, is it possible to hit the mass with another point particle? No, it isn't. Your aim would have to be perfectly good. In reality, however, there will always be some lateral component of the motion of the projectile (i.e., there will always be nonzero angular momentum). This will cause the projectile to deviate more and more from a radial trajectory as it gets closer; this is a consequence of the u_ϕ^2/r^3 term in the radial equation of motion. In practice, for compact objects and often even for normal stars, the angular momentum of matter is sufficient to ensure that there is not direct radial accretion. Of course, along the axis of rotation there is less centrifugal support, so accretion tends to form a disklike structure (the accretion disk).

Now, imagine that you have lots of such projectiles moving around the central mass. If they don't interact, their orbits will be unaffected by the presence of other projectiles. But if in reality the "projectiles" are streams of gas, they will collide with each other. This will tend to circularize the motion. But will anything happen once the motion is circularized?

Disks usually rotate such that each fluid element is moving almost (but not exactly!) in a circular orbit. If there were no interactions between fluid elements, **Ask class:** what would the angular velocity be as a function of radius? $\Omega \propto R^{-3/2}$, so there is a shearing flow. This means that coupling between adjacent radii exerts a force. **Ask class:** given that the outer parts rotate more slowly, in which direction will the force be and what will be the effect on the angular momentum and on the movement of mass? The inner part tries to speed up the outer part, which increases the velocity of the outer part. This increases the angular momentum of the outer part and decreases the angular momentum of the inner part, so the net result is that angular momentum is transferred outward and mass flows inward (some subtleties, of course). The disk spreads as a result. **Mention:** this has similarities to the effect of "shepherd moons" except there the coupling is purely gravitational.

Thus gas moving toward a massive object has a tendency to circularize, form a disk, and spread inward and outward. This is an "accretion disk". If the massive object has a surface, then often the matter spirals in until it hits the surface or interacts with the stellar magnetic field, whichever comes first. But a black hole has neither a surface nor a magnetic field. However, the strong-gravity effects of general relativity mean that the gas can't spiral all the way to the horizon, either. This is because of the ISCO, which we have discussed before. Particles spiraling inwards will release little energy inside the ISCO, so the efficiency

is just the binding energy in nearly circular orbits there. This is 6% of mc^2 for a nonrotating black hole, and up to 42% for a maximally rotating black hole (or 40% when you take into account that some of the radiation will be captured by the hole; this correction is negligible for slowly to moderately rotating holes).

Therefore, the accretion efficiency for black holes can in principle be the highest efficiency in astrophysics. Unlike stars, for black holes all the emitted energy must come from the accretion disk. We will therefore take a closer look at accretion disks.

One model of disks, which has many advantages (e.g., it is robust and does not depend on too many parameters) is one in which the disks are geometrically thin but optically thick. Let's think about the conditions for this to occur.

First, we make the assumption (standard for *all* models of accretion disks) that the disk itself has negligible mass compared to the central object. One consequence of this is that the gravity in the disk is dominated by the gravity of the central object, rather than the gas in the disk itself. On the other hand, the pressure forces within the disk are not necessarily negligible. Quantifying this, **Ask class:** what is the equation of hydrostatic equilibrium? In general it is $\nabla P = -\rho g$, where g is the local acceleration of gravity. We make another standard assumption, which is that the gas is orbiting in almost Keplerian orbits. That means that we can focus on the z component (normal to the disk plane) of the hydrostatic equation. If the central object has mass M and the fluid element of interest is an angle θ out of the plane, then the equation becomes

$$\frac{dP}{dz} = -\rho \frac{GM \sin\theta}{r^2} \ . \tag{1}$$

Let's call H the half-thickness of the disk; $H \ll r$ for a thin disk. To rough accuracy, $dP/dz = -P/H$ and $\sin\theta = H/r$. Then

$$\frac{P}{H} \approx \rho H/r^2 \left(\frac{GM}{r}\right) \Rightarrow \frac{P}{\rho} = c_s^2 \approx \left(\frac{H}{r}\right)^2 (GM/r) \ . \tag{2}$$

Since $v_K^2 = GM/r$, this means that the sound speed is $c_s \approx (H/r) v_K$. Therefore, the thin disk condition $H/r \ll 1$ implies (and is implied by) the condition that the sound speed is much less than the orbital speed. The sound speed increases with temperature, which increases with luminosity, which increases with accretion rate, so here we have an early warning that at high enough accretion rates the thin disk approximation is likely to break down. "High enough" turns out to mean near the Eddington luminosity. This is not a surprise, because near the Eddington luminosity radiation forces are strong enough to significantly modify the behavior of matter, and so having them puff up the disk is reasonable.

Temperature distribution of a thin disk

Now let's take a first stab at what the temperature distribution of a thin disk should

be. In a moment we'll do things more carefully, and find a surprising factor of 3.

Ask class: suppose that as each fluid element moves inward that it releases its energy locally, and that its energy is all gravitational. How much energy would an element of mass m release in going from a circular orbit at radius $r + dr$ to one at radius r? Gravitational potential energy is $E_g = -GMm/2r$, so the energy released is $GMmdr/2r^2$. **Warning: presaging** here is where the mysterious factor of 3 comes in. It turns out that in reality, far from the inner edge of a disk, the local energy released is a factor of 3 greater than that, due to viscous stresses associated with the transport of angular momentum. However, let us now focus on just the radial dependence, writing $dE_g \sim GMmdr/r^2$. That means that the luminosity of this annulus, for an accretion rate $\dot{m}$, is $dL \sim GM\dot{m}dr/r^2$. **Ask class:** what is the temperature, assuming the annulus radiates its energy as a blackbody? For a blackbody, $L = \sigma A T^4$. The area of the annulus is $2\pi r dr$, and since $L \sim M\dot{m}dr/r^2$ we have $T^4 \sim M\dot{m}r^{-3}$, or

$$T \sim \left(\frac{M\dot{m}}{r^3}\right)^{1/4} . \tag{3}$$

Therefore, the temperature increases as the fluid moves in. Another point is that from this equation we can see general scalings with the mass M of a central black hole. **Ask class:** suppose we have two nonrotating black holes, of mass M_1 and M_2, both accreting at the Eddington rate. What is the scaling of the temperature at, say, $r = 10\,M$ with the mass? The effects of general relativity depend on r/M, so suppose that we are interested in $r = xM$ (for example, $x = 6G/c^2$ for the innermost stable orbit). Also, as we saw in Lecture 3, the Eddington limiting luminosity scales with M, so suppose that the luminosity is $L = \epsilon L_E$, implying that $\dot{m} \propto \epsilon M$. The temperature is then $T \propto (M\epsilon M/(xM)^3)^{1/4} \sim M^{-1/4}$. This shows that as black holes get bigger, emission from their accretion disks get cooler, all else being equal. For example, a stellar-mass black hole accreting at nearly the Eddington rate has an inner disk temperature near 10^7 K, but a supermassive 10^8 K black hole accreting near Eddington has only a 10^5 K temperature.

Thin disks; more careful treatment

The simple treatment above neglects one very important point: if angular momentum is transported outwards, energy is as well. That means, qualitatively, that some of the gravitational energy released in the inner regions emerges as luminosity only in the outer regions. To derive this, we'll use an approach in some of Roger Blandford's notes, focusing on three conserved quantities: rest mass, angular momentum, and energy.

First, we assume the equation of continuity: the mass accretion rate is constant as a function of radius, so

$$\dot{M} = 2\pi r \Sigma v_r = \text{const} \tag{4}$$

where Σ is the surface density and v_r is the inward radial velocity.

Second, we treat angular momentum conservation. Assume for simplicity that the radial velocity is small and that the Newtonian form for angular momentum holds. Assume also that there is an inner radius r_I to the disk, and that no more angular momentum is lost inside that (for example, this might be thought a reasonable approximation at the ISCO). Then angular momentum conservation implies that the torque exerted by the disk inside radius r on the disk outside that radius is

$$G = \dot{M}\left[(Mr)^{1/2} - (Mr_I)^{1/2}\right] \ . \tag{5}$$

Third, energy conservation. The release of gravitational binding energy per unit time is, as we already saw, $\dot{E}_g = -\dot{M}d(m/2r)$. In addition there is a term due to viscous interactions. At a radius r where the angular velocity is $\Omega = (M/r^3)^{1/2}$, the rate of work done on the inner surface of an annulus is $-G\Omega$, and the net energy per time deposited in a ring is $\dot{E}_v = -d(G\Omega)$. The sum of the two is the luminosity released in the ring, $dL = \dot{E}_g + \dot{E}_v$. Evaluating this and replacing the Newtonian constant G we have

$$\frac{dL}{dr} = \frac{3G\dot{M}M}{2r^2}\left[1 - \left(\frac{r_I}{r}\right)^{1/2}\right] \ . \tag{6}$$

Whoa! Hold on here! This is different than what we might have expected. Far away from the inner edge r_I, this means that the local energy dissipation rate is *three times* the local release of gravitational energy. Where is the extra energy coming from? If we integrate $L(r)$ over the whole disk, we find that it gives $G\dot{M}m/2r_I$, as expected if the matter ends up in a circular orbit at radius r_I. However, close to r_I the energy dissipation rate is *less* than the local gravitational release. Therefore, what is happening is that matter near the inner part of the disk mas much of its energy going into transport of angular momentum rather than release of energy, and the extra energy is released further out. This factor of three was missed at first, but was pointed out by Kip Thorne.

Note, by the way, that this expression does not include any discussion of the mechanism by which angular momentum is transported.

Origin of angular momentum transport

The Reynolds number is defined as $R = VL/\nu$, where V is a typical velocity in a fluid system, L is a typical dimension, and ν is the kinematic viscosity. When $R \gg 1$ the fluid motion is turbulent. The kinematic viscosity is approximately $\nu = c_s\lambda$, where c_s is the sound speed and λ is a particle mean free path, so $R = (V/c_s)(L/\lambda)$. In accretion disks, the density is usually of order 1 g cm^{-3} (plus or minus a couple orders of magnitude), so the mean free path λ is much smaller than the system size L. Also, as we found earlier, the velocity is much greater than the sound speed for a geometrically thin disk, so $V \gg c_s$. Then $R \gg 1$ and the fluid is expected to flow turbulently. That means that the viscosity is a turbulent viscosity.

Calculations of microscopic viscosity indicate that this is much too small to transport the required angular momentum. In fact, it is misleading in a number of ways to talk about "viscosity" in this context, although it is common in the literature: such wording suggests a smooth viscous process, whereas the turbulence of the system is anything but smooth.

Magnetic fields to the rescue! In particular, the magnetorotational instability, by which weak magnetic fields are amplified by differential rotation, gives the required angular momentum transport. To understand the general principles, imagine that we have two masses m_1 and m_2 in circular orbits at radii r_1 and $r_2 > r_1$ around a mass M, and connected to each other by a spring with spring constant k. The mass m_1 moves with a higher angular frequency than the mass m_2. Therefore, as it moves, it exerts a force on m_2 that increases the angular momentum of m_2, and similarly m_2 retards the motion of m_1 and reduces its angular momentum. What happens next depends on the strength of the spring. If the spring is extremely strong (e.g., if we imagine it actually to be a rigid rod), then the two move along in a sort of compromise orbit, but nothing else would happen in particular. If instead the spring is weak, a surprising effect occurs. The increase in angular momentum of m_2 means that its orbital radius increases, and similarly the orbital radius of m_1 decreases. Their angular velocities are therefore even more different than they had been before, so the process runs away. One can show that if the angular velocity at some radius is Ω_0 and $m_2 \ll m_1$ then the dispersion relation leads to a characteristic equation (i.e., one in which a perturbation evolves as $\propto e^{\alpha t}$)

$$\alpha^4 + \left(\Omega_0^2 + 2k/m_2\right)\alpha^2 - (k/m_2)\left(3\Omega_0^2 - k/m_2\right) = 0\ . \tag{7}$$

It is left as the dreaded "exercise for the student" to determine the condition on k needed to ensure a runaway solution as opposed to an oscillatory one.

That's the magnetorotational instability. For *weak* magnetic fields in a disk with reasonable ionization, the field is stretched and amplified, in fact in a turbulent way, and the result is that angular momentum is transported outward and matter is transported inward. **Ask class:** what effects might stop the growth of the field? Reconnection, i.e., geometric reorganization of the field, does the job, as would other forms of dissipation.

As we said before, it is common in the literature to simply parameterize the angular momentum transport by writing it as an anomalous viscosity $\nu = \alpha c_s H$, where H is the scale height of the disk (this was done by Shakura and Sunyaev). Here α is a dimensionless constant, less than but comparable to unity, and the disks are called "alpha disks" as a result. The major physical effect that this does not include is turbulence, but many applications can be modeled nicely with this simplification.

Caveats and thick disks

Before we depart this subject, we need to offer some caveats. First, note that for high

luminosities, with puffed-up disks, the thin disk approximation is no longer good. Second, there has been a lot of work that shows that even at low accretion rates there is another solution: a geometrically thick, optically thin disk. For example, the energy release of gas near the black hole is so large that (at least energetically) it can drive a large wind off of the outer portions of the disk, meaning that $\dot{m}$ need not be constant with radius. The best current guess is that the "geometrically thin disk" solution operates when the accretion rate is between $\sim 1\%$ of Eddington and maybe several tens of percent of Eddington, whereas the geometrically thick, optically thin disk solution operates at lower accretion rates. Keep alert!

Additional references: the classic reference for accretion disks is "Accretion Processes in Astrophysics" by Frank, King, and Raine.