Neutron Star Structure

We now enter the study of neutron stars. Like black holes, neutron stars are one of the three possible endpoints of stellar evolution (the remaining one being white dwarfs). Also like black holes, neutron stars are very compact objects, so GR is important in their description. Unlike black holes, they have surfaces instead of horizons, so they are a lot more complicated than black holes. We'll start with an overall description of neutron stars, then discuss their high densities and strong magnetic fields.

Summary of Neutron Stars

A typical neutron star has a mass of $1.5 - 2\,M_\odot$ and a radius of $\sim 10 - 15$ km. It can have a spin frequency up to $\sim$1 kHz and a magnetic field up to perhaps 10^{15} G or more. Its surface gravity is around $2 - 3 \times 10^{14}$ cm s^{-2}, so mountains of even perfect crystals can't be higher than < 1 mm, meaning that these are the smoothest surfaces in the universe. They have many types of behavior, including pulsing (in radio, IR, opt, UV, X-ray, and gamma-rays, but this is rarely all seen from a single object), glitching, accreting, and possibly gravitational wave emission. They are the best clocks in the universe; the most stable are a million times more stable in the short term than the best atomic clocks. Their cores are at several times nuclear density, and may be composed of exotic matter such as quark-gluon plasmas, strange matter, kaon condensates, or other weird stuff. In their interior they are superconducting and superfluid, with transition temperatures around a hundred million degrees Kelvin. All these extremes mean that neutron stars are attractive to study for people who want to push the envelope of fundamental theories about gravity, magnetic fields, and high-density matter.

High densities

Let's start, then, with high densities. An essential new concept that is introduced in high densities is *Fermi energy*. The easiest way to think about this is in terms of the uncertainty principle,

$$\Delta p \Delta x > \hbar \ . \tag{1}$$

For you sticklers for accuracy: yes, the actual uncertainty principle is $\Delta p \Delta x \geq \hbar/2$, but here we're not worried about exact factors. If something is localized to a region of size Δx, then its momentum must be at least $\hbar/\Delta x$. That means that in a dense environment, there is a momentum, and hence an energy, associated with the confinement. Therefore, squeezing something increases its total energy, and this Fermi energy acts as a pressure (sometimes called degeneracy pressure). The existence of this energy has a profound role in the structure of white dwarfs and neutron stars. For example, unlike normal stars, which get larger as they get more massive, white dwarfs are *smaller* at higher masses.

Let's start with some basic numbers. If the energy and momentum are nonrelativistic, then the Fermi energy E_F is related to the Fermi momentum $p_F \sim \hbar/\Delta x$ by $E_F \approx p_F^2/2m$, where m is the rest mass of the particle. Since $\Delta x \sim n^{-1/3}$, where n is the number density of the particle, in this regime $E_F \sim n^{2/3}$. At some point, however, $E_F > mc^2$. Then $E_F \sim p_F c$, so $E_F \sim n^{1/3}$. For electrons, the crossover to relativistic Fermi energy happens at a density $\rho \sim 10^6$ g cm^{-3}, assuming a fully ionized plasma with two nucleons per electron. For protons and neutrons the crossover is at about 6×10^{15} g cm^{-3} (it scales as the particle's mass cubed). The maximum density in neutron stars is no more than 10^{15} g cm^{-3}, so for most of the mass electrons are highly relativistic but neutrons and protons are at best mildly relativistic.

Let's now think about what that means. **Ask class:** suppose we have matter in which electrons, protons, and neutrons all have the same number density. For a low density, which has the highest Fermi energy? The electrons, since at low densities the Fermi energy goes like the inverse of the particle mass. **Ask class:** given what we said before, what is the approximate value of the electron Fermi energy when $\rho = 10^6$ g cm^{-3}? That's the relativistic transition, so $E_F \approx m_e c^2 \approx 0.5$ MeV. Then at 10^7 g cm^{-3} the Fermi energy is about 1 MeV, and each factor of 10 doubles the Fermi energy since $E_F \sim n^{1/3}$ in the relativistic regime. What that means is that the energetic "cost" of adding another electron to the system is not just $m_e c^2$, as it would be normally, but is $m_e c^2 + E_F$. It therefore becomes less and less favorable to have electrons around as the density increases.

Now, in free space neutrons are unstable. This is because the sum of the masses of an electron and a proton is about 1.5 MeV short of the mass of a neutron, so it is energetically favorable to decay. **Ask class:** what happens, though, at high density? If $m_p + m_e + E_F > m_n$, then it is energetically favorable to combine a proton and an electron into a neutron. Therefore, at higher densities matter becomes more and more neutron-rich. First, atoms get more neutrons, so you get nuclei such as ^{120}Rb, with 40 protons and 80 neutrons. Then, at about 4×10^{11} g cm^{-3} it becomes favorable to have free neutrons floating around, along with some nuclei (this is called "neutron drip" because the effect is that neutrons drip out of the nuclei). At even higher densities, the matter is essentially a smooth distribution of neutrons plus a $\sim 5 - 10\%$ smattering of protons and electrons. At higher densities yet (here we're talking about nearly 10^{15} g cm^{-3}), the neutron Fermi energy could become high enough that it is favorable to have other particles appear.

It is currently unknown whether such particles will appear, and this is a focus of much present-day research. If they do, it means that the energetic "cost" of going to higher density is less than it would be otherwise, since energy is released by the appearance of other, exotic particles instead of more neutrons. In turn, this means that it is easier to compress the star: squashing it a bit doesn't raise the energy as much as you would have thought. Another way of saying this is that when a density-induced phase transition occurs (here, a transition to other types of particles), the equation of state is "soft".

Well, that means that it can't support as much mass. That's because as more mass is added, the star compresses more and more, so its gravitational compression increases. If pressure doesn't increase to compensate, in it goes and forms a black hole. What all this means is that by measuring the mass and radius of a neutron star, or by establishing the maximum mass of a neutron star, one gets valuable information about the equation of state, and hence about nuclear physics at very high density. This is just one of many ways in which study of neutron stars has direct implications for microphysics.

Comment: the extra "squishiness" of matter when it is near a density-induced phase transition may also have importance in the early universe. It's been pointed out that when the universe goes from being a quark-gluon plasma to being made of nucleons (at about 10^{-5} s after the Big Bang), this is a density-induced phase transition, so it may be comparatively easy to form black holes then. Perhaps this led to the formation of so many black holes that they form dark matter; incidentally, because this event happened before big bang nucleosynthesis, no baryon number constraints are violated. This is not the leading model for dark matter, but it is thought-provoking.

Big Honking Magnetic Fields

In addition to ultrahigh densities, another unique aspect of neutron stars is their magnetic fields. By a factor of at least 10^4, and possibly 10^7, neutron stars have the strongest magnetic fields in the known universe. The fields can therefore have extremely important effects on matter in ways not approached anywhere else. Here we'll concentrate on "ordinary" fields of $\sim 10^{12}$ G, and will postpone discussion of higher fields to the special lecture on magnetars.

As we have discussed, magnetic fields tend to have a minor impact on the structure of ordinary stars, and even a minor impact on their spectra (fields less than 10^5 G or so are difficult to detect in stars other than the Sun). For neutron stars, however, their effect on the spectrum is dominant, as is their effect on radiation transport properties, which are in turn the most important energy transport mechanisms near the surface of the star.

Let's start the treatment by thinking of a free electron spiraling in a magnetic field. Suppose that it is moving at an angular frequency ω in a circle of radius r, and apply the classical force balance equation. **Ask class:** what is the centrifugal force for an electron, of mass m_e? It's just $m_e\omega^2 r$. **Ask class:** if it is moving at an azimuthal velocity v_ϕ, what is the magnetic force on the electron for a magnetic field B? Simply eBv_ϕ/c. Equating the two, and using $v_\phi = \omega r$, we find the frequency (the cyclotron frequency)

$$\omega_c = \frac{eB}{m_e c} = 11.5\,\text{keV}\,B_{12}\;,\qquad(2)$$

where as usual the convention is that $B_{12} = B/10^{12}$ G. Another important quantity is the

scale length of the radius of the orbit, which is given by the Bohr-Sommerfeld quantization rule:

$$L_z = m_e \omega r^2 \sim m\hbar \tag{3}$$

where $m = \pm 1, \pm 2, \ldots$ is the azimuthal quantum number. Therefore, the radius of the mth orbital is something like

$$r_m \sim \hat{r} \sqrt{|m|} \,, \tag{4}$$

where $\hat{r} = (\hbar c / eB)^{1/2} = 2.5 \times 10^{-10} B_{12}^{-1/2}$ cm is the Landau radius. More exactly, $r_m = \hat{r}\sqrt{2|m|+1}$.

Ask class: we now want to estimate when the orbitals of atoms are affected significantly by the magnetic field. What should we compare? We could compare energies (cyclotron versus Coulomb), forces (magnetic versus Coulomb), or distances (size of atomic orbital versus radius of mth Landau orbital). All three give approximately the same answer. Using forces, equality happens when

$$\frac{Ze^2}{r_m^2} = \frac{ev_m B}{c} = \frac{e\omega_c r_m B}{c} \,, \tag{5}$$

giving the critical field for state m of a hydrogenic atom of nuclear charge Z as

$$B_c = \frac{Z^2}{(2m+1)^3} B_0 \,. \tag{6}$$

The critical field for the ground state of hydrogen is $B_0 = m_e^2 c e^3 / \hbar^3 = 2.35 \times 10^9$ G, at which point the cyclotron energy is 2×13.6 eV. **Ask class:** given this, are magnetic effects likely to be most important for ground states or excited states? For hydrogen or for heavier atoms? Excited states and low-charge nuclei are easiest to get in the magnetically dominated regime. In fact, lab experiments with high Rydberg level hydrogen ($n \sim 100 - 200$!) have shown some of these effects.

Ask class: consider now a neutron star with a magnetic field $\sim 10^{12}$ G. The energy difference between the ground state and first excited Landau state is $\hbar\omega_c$. At what approximate surface temperature would one expect excited Landau states to exist? At about $kT = 10$ keV, or $T \sim 10^8$ K. This is much hotter than most neutron star surfaces, which have $T \sim 10^5 - 10^6$ K.

When $B \gg B_c$ and $kT \ll \hbar\omega_c$, atoms become essentially cylindrical (and are simple to treat when B is very large, but complicated when B is intermediate). In this limit, the Coulomb force dominates only along the magnetic field, and the orbitals are tightly bound across the field. The length across the field for the ground state is therefore $\hat{r}$, and the length along the field is greater, with some length ℓ to be determined. The energy of these orbitals is

$$E \sim \frac{\hbar^2}{2m_e \ell^2} - \frac{Ze^2}{\ell} \ln(\ell/\hat{r}) \,. \tag{7}$$

Minimizing with respect to ℓ gives

$$\ell \sim \left(\frac{a_0/Z\hat{r}}{\ln(a_0/Z\hat{r})} \right) \hat{r} \tag{8}$$

where $a_0 = 5 \times 10^{-9}$ cm is the Bohr radius. For example, when $B = 10^{12}$ G, $a_0 \approx 20\hat{r}$, so $\ell \approx 7\hat{r}$ for hydrogen. The energy is then

$$E \sim Z^2 \frac{\hbar^2}{m_e a_0^2} \ln^2 \left(a_0/Z\hat{r} \right) \ . \tag{9}$$

The factor before the log is just the usual ground state energy without any magnetic field. For $B = 10^{12}$ G, the multiplying factor in this formula is about 9, predicting about 120 eV for the ground state energy of hydrogen. The real value is about 160 eV.

This is amazing! It means that for a typical NS magnetic field, the ground state energy is *ten times* what it is for no field! This makes a huge difference in many ways. For example, the ionization equation (the Saha equation, for those who have taken Radiative Processes) has to be modified in strong fields in other ways, but if we applied it blindly for no magnetic field in a neutron star atmosphere we'd find it predicted no neutral hydrogen. But the binding energy is so huge that hydrogen can exist at even $\sim 10^6$ K in the neutral state. This means that there is a significant extra source of opacity here.

Superconductivity and Superfluidity

One of the principles that we just encountered is that nature, being lazy, will go for the lowest energy state possible in a given circumstance. That's why there is progressive neutronization of matter at higher and higher densities: it's a lower energy state. In that same general spirit, we also can have superconductivity and superfluidity in neutron stars.

The general idea is that if there is an attractive pairing interaction between fermions, they can couple to form a state with integral spin, and therefore can act like bosons. At a low enough temperature, these "bosons" can form a condensate-like state in which all of the bosons occupy the same quantum state and form a superfluid. If the component fermions are charged, this forms a superconductor. In normal laboratory experience, the pairing is electronic and happens only at very low temperatures (other than the ceramic high T_c superconductors, which do their thing at liquid nitrogen temperatures or a bit above, almost all laboratory superfluid or superconducting phenomena are observed at temperatures less than 20 K). However, in the dense core of neutron stars, nucleonic pairing can happen.

As with all highly degenerate systems, pairing occurs between states near the Fermi surface (recall that in the cores of NS, both protons and neutrons are degenerate, just not relativistically so). Since there are many more neutrons than protons, neutrons and protons can't pair up easily because their momenta are substantially different. We can

therefore consider only $n - n$ and $p - p$ pairings. The first gives a superfluid, the second a superconductor. The transition temperatures are extremely difficult to compute from first principles, but the guess is that they are in the range of a hundred million to a billion K.

What are the effects of superconductivity and superfluidity? The evidence is pretty indirect, but superfluidity has been invoked to explain glitches (sudden but small changes in the spin frequency of pulsars) and the evolution of the temperature and magnetic fields of neutron stars. We'll also encounter superfluidity in magnetars, where it might have an impact on quasi-periodic oscillations seen in especially large bursts.