Frame-dragging: occurs generally around rotating objects. It's just that the exact Kerr form only applies to black holes.

See Longair 16.3 Also see Novikov+Thorne contribution to the "Black Holes" 1972 proceedings, ed. de Witt and de Witt (our library has it).

Accretion Disks

Angular momentum

Now let's go back to black holes. Black holes being what they are, something that falls into one disappears without a peep. It might therefore seem that accretion onto a black hole would release no energy. It isn't the case, however, and the reason has to do with angular momentum.

Thinking in terms of Newtonian gravity, suppose you have a point source of mass. **Ask class:** taking gravity into effect, is it possible to hit the mass with another point particle? No, it isn't. Your aim would have to be perfectly good. In reality, however, there will always be some lateral component of the motion of the projectile (i.e., there will always be nonzero angular momentum). This will cause the projectile to deviate more and more from a radial trajectory as it gets closer; this is a consequence of the ℓ^2/r^3 term in the radial equation of motion. In practice, for compact objects and often even for normal stars, the angular momentum of matter is sufficient to ensure that there is not direct radial accretion. Of course, along the axis of rotation there is less centrifugal support, so this tends to form a disklike structure (the accretion disk).

Now, imagine that you have lots of such projectiles moving around the central mass. If they don't interact, their orbits will be unaffected by the presence of other projectiles. But if in reality the "projectiles" are streams of gas, they will collide with each other. This will tend to circularize the motion. But will anything happen once the motion is circularized?

Disks usually rotate such that each fluid element is moving almost (but not exactly!) in a circular orbit. If there were no interactions between fluid elements, **Ask class:** what would the angular velocity be as a function of radius? $\Omega \propto R^{-3/2}$, so there is a shearing flow. This means that coupling between adjacent radii exerts a force. **Ask class:** given that the outer parts rotate more slowly, in which direction will the force be and what will be the effect on the angular momentum and on the movement of mass? Inner tries to speed up outer, giving it a higher velocity. This increases the angular momentum of the outer, decreases the angular momentum of the inner, so net result is that angular momentum is transferred outwards and mass flows inwards (some subtleties, of course). The disk spreads as a result. **Mention:** this has similarities to the effect of "shepherd moons" except there the coupling is purely gravitational.

So, gas moving towards a massive object has a tendency to circularize, form a disk, and spread inward and outward. This is an "accretion disk". If the massive object has a surface, then often the matter spirals in until it hits the surface or interacts with the stellar magnetic field, whichever comes first. But a black hole has neither a surface nor a magnetic field. However, strong-gravity effects of general relativity mean that the gas can't spiral all the way to the horizon, either. This is because of the ISCO, which we have discussed before. Particles spiraling inwards will release little energy inside the ISCO, so the efficiency is just the binding energy in nearly circular orbits there. This is 6% of mc^2 for a nonrotating black hole, and up to 42% for a maximally rotating black hole.

Therefore, the accretion efficiency for black holes can in principle be the highest accretion efficiencies in astrophysics. Unlike stars, for black holes all the emitted energy must come from the accretion disk. We will therefore take a closer look at accretion disks.

Now we'll think more carefully about accretion disks themselves. One model of disks, which has many advantages (e.g., it is robust and does not depend on too many parameters) is one in which the disks are geometrically thin but optically thick. Let's think about the conditions for this to occur.

First, we make the assumption (standard for *all* models of accretion disks) that the disk itself has negligible mass compared to the central object. One aspect of that is that the gravity in the disk is dominated by the gravity of the central object, not the gas in the disk itself. On the other hand, pressure forces within the disk are not necessarily negligible. Quantifying this, **Ask class:** what is the equation of hydrostatic equilibrium? In general it is $\nabla P = -\rho g$, where g is the local acceleration of gravity. We make another standard assumption, which is that the gas is orbiting in almost Keplerian orbits. That means that we can focus on the z component (normal to the disk plane) of the hydrostatic equation. If the central object has mass M and the fluid element of interest is an angle θ out of the plane, then (**draw diagram**) the equation becomes

$$\frac{dP}{dz} = -\rho\frac{GM\sin\theta}{r^2}\ . \tag{1}$$

Let's call H the half-thickness of the disk; $H \ll r$ for a thin disk. To rough accuracy, $dP/dz = -P/H$ and $\sin\theta = H/r$. Then

$$\frac{P}{H} \approx \rho H/r^2\left(\frac{GM}{r}\right) \Rightarrow \frac{P}{\rho} = c_s^2 \approx \left(\frac{H}{r}\right)^2 (GM/r)\ . \tag{2}$$

Since $v_K^2 = GM/r$, this means that the sound speed is $c_s \approx (H/r)v_K$. Therefore, the thin disk condition $H/r \ll 1$ implies (and is implied by) the condition that the sound speed is much less than the orbital speed. The sound speed increases with temperature, which increases with luminosity, which increases with accretion rate, so here we have an early warning that at high enough accretion rates the thin disk approximation is likely to break down. "High enough" turns out to mean near the Eddington luminosity. This is not a surprise, because near the Eddington luminosity radiation forces are strong enough to significantly modify the behavior of matter, and so having them puff up the disk is reasonable.

Temperature and frequency distribution of a thin disk

Now let's take a first stab at what the temperature and frequency distribution of a thin disk should be. In a moment we'll do things more carefully, and find a surprising factor of 3. Longair section 16.3 has a different but also careful derivation.

Ask class: suppose that as each fluid element moves inward that it releases its energy locally, and that its energy is all gravitational. How much energy would an element of mass m release in going from a circular orbit at radius $r+dr$ to one at radius r? Gravitational potential energy is $E_g = -GMm/2r$, so the energy released is $GMmdr/2r^2$. **Warning: presaging** here is where the mysterious factor of 3 comes in. It turns out that in reality, far from the inner edge of a disk, the local energy released is a factor of 3 greater than that, due to viscous stresses associated with the transport of angular momentum. However, let us now focus on just the radial dependence, writing $dE_g \sim GMmdr/r^2$. That means that the luminosity of this annulus, for an accretion rate $\dot{m}$, is $dL \sim GM\dot{m}dr/r^2$. **Ask class:** what is the temperature, assuming the annulus radiates its energy as a blackbody? For a blackbody, $L = \sigma AT^4$. The area of the annulus is $2\pi rdr$, and since $L \sim M\dot{m}dr/r^2$ we have $T^4 \sim M\dot{m}r^{-3}$, or

$$T \sim \left(\frac{M\dot{m}}{r^3}\right)^{1/4} . \tag{3}$$

Therefore, the temperature increases as the fluid moves in. Another point is that from this equation we can see general scalings with the mass M of a central black hole. **Ask class:** suppose we have two nonrotating black holes, of mass M_1 and M_2, both accreting at the Eddington rate. What is the scaling of the temperature at, say, $r = 10\,M$ with the mass? The effects of general relativity depend on r/M, so suppose that we are interested in $r = xM$ (for example, $x = 6G/c^2$ for the innermost stable orbit). Also, as we saw last lecture, the Eddington limiting luminosity scales with M, so suppose that the luminosity is $L = \epsilon L_E$, implying that $\dot{m} \propto \epsilon M$. The temperature is then $T \propto (M\epsilon M/(xM)^3)^{1/4} \sim M^{-1/4}$. This shows that as black holes get bigger, emission from their accretion disks get cooler, all else being equal. For example, a stellar-mass black hole accreting at nearly the Eddington rate has an inner disk temperature near 10^7 K, but a supermassive 10^8 K black hole accreting near Eddington has only a 10^5 K temperature.

Continuing with our simple model, we can also derive the emission spectrum in a range of frequencies. Because "thin disks" are assumed to be optically thick (the "thin" is geometrical, remember), we can assume that in each annulus they radiate as blackbodies with the local temperature $T(r)$. The emission spectrum is this integrated over the whole disk, which we assume goes from an inner radius r_I to a maximum radius $r_{\rm max}$. Then

$$I(\nu) \propto \int_{r_I}^{r_{\rm max}} 2\pi r B[T(r),\nu]dr \tag{4}$$

where $B(T,r)$ is the Planck function $B(T,r) \propto \nu^3[\exp(h\nu/kT)-1]^{-1}$. In the region where $T \propto r^{-3/4}$ we can change variables, $dr \propto (1/T)^{1/3}d(1/T)$. Then

$$I(\nu) \propto \int_{1/T_I}^{1/T_{\rm max}} \left(\frac{1}{T}\right)^{4/3} \nu^3 \left[\exp\left(\frac{h\nu}{kT}\right)-1\right]^{-1} \left(\frac{1}{T}\right)^{1/3} d\left(\frac{1}{T}\right) \ . \tag{5}$$

If we now make another change of variables, this time to $x = h\nu/kT$, then

$$I(\nu) \propto \frac{\nu^3}{\nu^{8/3}} \int_{h\nu/T_I}^{h\nu/T_{\rm max}} x^{4/3}(\exp x - 1)^{-1}x^{1/3}dx \ . \tag{6}$$

In the middle portion of the spectrum, $h\nu/T_I \ll 1$ and $h\nu/T_{\rm max} \gg 1$, so we can approximate the limits as 0 and ∞, respectively. But the integral is then simply a definite integral, which does not depend on ν. Therefore, in the region of the spectrum with frequencies corresponding to between r_I and $r_{\rm max}$, the spectral emissivity simply goes as $I(\nu) \propto \nu^{1/3}$. **Ask class:** what is the emissivity at frequencies well below the frequency corresponding to $r_{\rm max}$? Below this frequency, emission from any of the annuli in the disk is in the Rayleigh-Jeans portion of the spectrum, $I(\nu) \sim \nu^2$, so their sum is as well and hence the emissivity below $h\nu = kT_{\rm outer}$ goes as $I(\nu) \propto \nu^2$. **Ask class:** what about at frequencies well above that corresponding to r_I? Above the thermal frequency of the inner edge, the spectrum falls off exponentially for all annuli. The spectrum is then dominated by the emission from the inner edge (the temperature is highest there, and the luminosity is greatest), so $I(\nu) \propto \exp(-h\nu/kT_{\rm inner})$.

Comment: the point of this exercise, and the continuation below, is to see how a simple model of an accretion disk may be solved. However, real life is (as always) more complicated. In particular, this spectral form is not a particularly good fit to data, and other components seem necessary (e.g., a hot corona above the disk that reprocesses radiation by Compton scattering). Also, near the hole one must include GR effects (different angular momentum and energy distributions, for example). A full GR treatment is given by Novikov and Thorne in the 1972 "Black Holes" proceedings (de Witt and de Witt, editors; our library has it). In practice, many people use a "pseudo-Newtonian" potential $\Phi = -GM/(r-2M)$, which mocks up many GR effects fairly well without slowing down code.

Thin disks; more careful treatment (Longair 16.3.3)

The simple treatment above neglects one very important point: if angular momentum is transported outwards, energy is as well. That means, qualitatively, that some of the gravitational energy released in the inner regions emerges as luminosity only in the outer regions. To derive this, we'll use an approach in some of Roger Blandford's notes, focusing on three conserved quantities: rest mass, angular momentum, and energy.

First, we assume the equation of continuity: the mass accretion rate is constant as a function of radius, so

$$\dot{M} = 2\pi r \Sigma v_r = \text{const} \tag{7}$$

where Σ is the surface density and v_r is the inward radial velocity.

Second, we treat angular momentum conservation. Assume for simplicity that the radial velocity is small and that the Newtonian form for angular momentum holds. Assume also that there is an inner radius r_I to the disk, and that no more angular momentum is lost inside that (for example, this might be thought a reasonable approximation at the ISCO). Then angular momentum conservation implies that the torque exerted by the disk inside radius r on the disk outside that radius is

$$G = \dot{M}\left[(Mr)^{1/2} - (Mr_I)^{1/2}\right] . \tag{8}$$

Third, energy conservation. The release of gravitational binding energy per unit time is, as we already saw, $\dot{E}_g = -\dot{M}d(m/2r)$. In addition there is a term due to viscous interactions. At a radius r where the angular velocity is $\Omega = (M/r^3)^{1/2}$, the rate of work done on the inner surface of an annulus is $-G\Omega$, and the net energy per time deposited in a ring is $\dot{E}_v = -d(G\Omega)$. The sum of the two is the luminosity released in the ring, $dL = \dot{E}_g + \dot{E}_v$. Evaluating this and replacing the Newtonian constant G we have

$$\frac{dL}{dr} = \frac{3G\dot{M}M}{2r^2}\left[1 - \left(\frac{r_I}{r}\right)^{1/2}\right] . \tag{9}$$

Whoa! Hold on here! This is different than what we might have expected. Far away from the inner edge r_I, this means that the local energy dissipation rate is *three times* the local release of gravitational energy. Where is the extra energy coming from? If we integrate $L(r)$ over the whole disk, we find that it gives $G\dot{M}m/2r_I$, as expected if the matter ends up in a circular orbit at radius r_I. However, close to r_I the energy dissipation rate is *less* than the local gravitational release. Therefore, what is happening is that matter near the inner part of the disk mas much of its energy going into transport of angular momentum rather than release of energy, and the extra energy is released further out. This factor of three was missed at first, but was pointed out by Kip Thorne.

Note, by the way, that this expression does not include the viscosity. That's why it was possible to do the derivation without specifying the viscosity; Longair does introduce the viscosity, in part because it *is* important for other reasons (e.g., to get the radial velocity or surface density).

Origin of viscosity

The Reynolds number is defined as $R = VL/\nu$, where V is a typical velocity in a fluid system, L is a typical dimension, and ν is the kinematic viscosity. When $R \gg 1$ the fluid motion is turbulent. The kinematic viscosity is approximately $\nu = c_s\lambda$, where c_s is the sound speed and λ is a particle mean free path, so $R = (V/c_s)(L/\lambda)$. In accretion disks, the density is usually of order 1 g cm^{-3} (plus or minus a couple orders of magnitude), so the mean free path λ is much smaller than the system size L. Also, as we found earlier, the velocity is much greater than the sound speed for a geometrically thin disk, so $V \gg c_s$. Then $R \gg 1$ and the fluid is expected to flow turbulently. That means that the viscosity is a turbulent viscosity. Calculations of microscopic viscosity indicate that this is much too small. However, in the last decade it has been shown that magnetohydrodynamic effects can do it. In particular, the magnetorotational instability, by which weak magnetic fields are amplified by differential rotation, gives the required viscosity. This is a large subject. Jim Stone is one of the world's experts on this, and has comprehensive computer codes to treat viscosity and disk physics. However, it is common to simply parameterize the viscosity by writing it $\nu = \alpha c_s H$, where H is the scale height of the disk (this was done by Shakura and Sunyaev). Here α is a dimensionless constant, less than but comparable to unity, and the disks are called "alpha disks" as a result.

Caveats and thick disks

Before departing this subject, some caveats. First, note that for high luminosities, with puffed-up disks, the thin disk approximation is no longer good. Second, there has been a lot of work in the past decade showing that even at low accretion rates there is another solution: a geometrically thick, optically thin disk. In particular, the energy release of gas near the black hole is so large that (at least energetically) it can drive a large wind off of the outer portions of the disk, meaning that $\dot{m}$ is not constant with radius. Observations are inconclusive, and which solution actually operates when is an open question. Keep alert!