Overview of Gravitational Radiation

As we contemplate the triumphant direct detection of gravitational radiation, it is useful to consider what such detections will teach us about the universe. The first detection, GW150914, was of course of immediate significance because it was a direct confirmation of a dramatic prediction of general relativity: to paraphrase John Wheeler, that spacetime tells sources how to move, and moving sources tell spacetime how to ripple. The first double neutron star detection, GW170817, had even more to tell us, about gamma-ray bursts and the origin of heavy elements.

Beyond the initial detections, gravitational wave science will pass into the realm of astronomy, and will give us new observational windows onto some of the most dynamic phenomena in the universe. These include merging neutron stars and black holes, supernova explosions, and possibly echoes from the very early history of the universe as a whole. They have also already provided the cleanest tests of predictions of general relativity in the realm of strong gravity, with much more to come.

However, there are important differences from standard astronomy. In electromagnetic observations, in every waveband there are sources so strong that they can be detected even if you know nothing about the source. You don't need to understand nuclear fusion in order to see the Sun! In contrast, most of the expected sources of gravitational radiation are so weak that we expect that usually sophisticated statistical techniques will be required to detect them at all (with occasional happy exceptions such as GW150914 and GW170817, which were so strong that they could be seen by eye after moderate bandpass and notch filtering of the data). A standard technique involves matching templates of expected waveforms against the observed data stream. Maximum sensitivity therefore requires a certain understanding of what the sources look like, and thus of the characteristics of those sources. In addition, it will be important to put each detection into an astrophysical context so that the implications of the discoveries are evident.

As an aside, it is useful to remember that historically the most interesting sources discovered with a new telescope or satellite have often been unexpected. This is also possible with gravitational radiation. However, you can't sell a large project by appealing entirely to the unknown, so we should at least describe what we *can* imagine at this point!

Before discussing types of sources, though, we need to have some general perspective on how gravitational radiation is generated and how strong it is. We will begin by discussing radiation in a general context.

By definition, a radiation field must be able to carry energy to infinity. If the amplitude of the field a distance r from the source in the direction (θ, ϕ) is $A(r, \theta, \phi)$, the flux through a spherical surface at r is $F(r, \theta, \phi) \propto A^2(r, \theta, \phi)$. If for simplicity we assume that the radiation is spherically symmetric, $A(r, \theta, \phi) = A(r)$, this means that the luminosity at a distance r is $L(r) \propto A^2(r) 4\pi r^2$. Note, though, that when one expands the static field of a source in moments, the slowest-decreasing moment (the monopole) decreases like $A(r) \propto 1/r^2$, which implies that $L(r) \propto 1/r^2$ and hence no energy is carried to infinity. This tells us two things, regardless of the nature of the radiation (e.g., electromagnetic or gravitational). First, radiation requires time variation of the source. Second, the amplitude must scale as $1/r$ far from the source.

We can now explore what types of variation will produce radiation. We'll start with electromagnetic radiation, and expand in moments. Suppose that we are far from some distribution of electric charges, which could be in motion. For a charge density $\rho_e(\mathbf{r})$, the monopole moment is $\int \rho_e(\mathbf{r}) d^3r$. We assume that the volume over which we perform the integral encompasses the entire system; no charges can enter or leave. As a result, the monopole moment is simply the total charge Q, which cannot vary, so there is no electromagnetic monopolar radiation. The next static moment is the dipole moment, $\int \rho_e(\mathbf{r})\mathbf{r} d^3r$. There is no applicable conservation law, so electric dipole radiation is possible. One can also look at the variation of currents. The lowest order such variation (the "magnetic dipole") is $\int \rho_e(\mathbf{r})\mathbf{r} \times \mathbf{v}(\mathbf{r}) d^3r$. Once again this can vary, so magnetic dipole radiation is possible. The lower order moments will typically dominate the field unless their variation is reduced or eliminated by some special symmetry.

Now consider gravitational radiation. Let the mass-energy density be $\rho(\mathbf{r})$. The monopole moment is $\int \rho(\mathbf{r}) d^3r$, which is simply the total mass-energy. This is constant, so there cannot be monopolar gravitational radiation. The static dipole moment is $\int \rho(\mathbf{r})\mathbf{r} d^3r$. This, however, is just the center of mass-energy of the system. In the center of mass frame, therefore, this moment does not change, so there cannot be the equivalent of electric dipolar radiation in this frame (or any other, since the existence of radiation is frame-independent). The counterpart to the magnetic dipolar moment is $\int \rho(\mathbf{r})\mathbf{r} \times \mathbf{v}(\mathbf{r}) d^3r$. This, however, is simply the total angular momentum of the system, so its conservation means that there is no magnetic dipolar gravitational radiation either. The next static moment is quadrupolar: $I_{ij} = \int \rho(\mathbf{r}) r_i r_j d^3r$. This does not have to be conserved, and thus there can be quadrupolar gravitational radiation.

This allows us to draw general conclusions about the type of motion that can generate gravitational radiation. A spherically symmetric variation is only monopolar, so it does not produce radiation. No matter how violent an explosion (even a supernova!) or a collapse (even into a black hole!), no gravitational radiation is emitted if spherical symmetry is maintained. This is consistent with our understanding of Newtonian gravity: the gravitational field outside a spherically symmetric collection of matter is just what it would be if all that matter were concentrated in a point at the center, so we don't expect the gravitational field to change by spherically symmetric motion (the equivalent statement in general relativity is called Birkhoff's Theorem). This means that we do need to make one adjustment to our quadrupolar argument above. Note that a spherically symmetric expansion or contraction *will* change I_{ij}. Thus we need to change the expression: it turns out that when we add a component to I_{ij} that makes the sum traceless, then the combination is what needs to vary to produce gravitational radiation.

In addition, a rotation that preserves axisymmetry (without contraction or expansion) does not generate gravitational radiation because the quadrupolar and higher moments are unaltered. Therefore, for example, a neutron star can rotate arbitrarily rapidly without emitting gravitational radiation as long as it maintains stationarity and axisymmetry and rotates around the axis of symmetry.

This immediately allows us to focus on the most promising types of sources for gravitational wave emission. The general categories are: binaries, continuous wave sources (e.g., rotating stars with nonaxisymmetric lumps), bursts (e.g., asymmetric collapses), and stochastic sources (i.e., individually unresolved sources with random phases; the most interesting of these would be a background of gravitational waves from the early universe).

We can now make some order of magnitude estimates. What is the approximate expression for the dimensionless amplitude h of a metric perturbation, a distance r from a source? Note, by the way, that because gravitational waves are perturbations in spacetime, h is related to the fractional deviation of the spacetime from the Minkowski (flat) spacetime. Thus h is of the order of the fractional change in length induced by a passing gravitational wave, if the length in question is of order the gravitational wavelength.

We argued that the lowest order radiation has to be quadrupolar, and hence depends on the quadrupole moment I. This moment is $I_{ij} = \int \rho r_i r_j d^3x$ (plus the trace-free modification we mentioned), so it has dimensions MR^2, where M is some mass and R is a characteristic dimension. We also argued that the amplitude is proportional to $1/r$, so we have

$$h \sim MR^2/r \; . \tag{1}$$

We know that h is dimensionless, so how do we determine what else goes in here? In GR we usually set $G = c = 1$, which means that mass, distance, and time all have the same effective "units", but we can't, for example, turn a distance squared into a distance. Our current expression has effective units of distance squared (or mass squared, or time squared). We note that time derivatives have to be involved, since a static system can't emit anything. Two time derivatives will cancel out the current units, so we now have

$$h \sim \frac{1}{r}\frac{\partial^2(MR^2)}{\partial t^2} \; . \tag{2}$$

Now what? To get back to physical units we have to restore factors of G and c. It is useful to remember certain conversions: for example, if M is a mass, GM/c^2 has units of distance, and GM/c^3 has units of time. Playing with this for a while gives finally

$$h \sim \frac{G}{c^4}\frac{1}{r}\frac{\partial^2(MR^2)}{\partial t^2}\ . \tag{3}$$

Since G is small and c is large, the prefactor is *tiny*! That tells us that unless MR^2 is large, the system is changing fast, and r is small, the metric perturbation is minuscule.

Let's make a very rough estimate for a circular binary. Suppose the total mass is $M = m_1 + m_2$, the reduced mass is $\mu = m_1 m_2/M$, and the semimajor axis is a, so the orbital frequency Ω is given by $\Omega^2 a^3 = GM$. Without worrying about precise factors, we say that $\partial^2/\partial t^2 \sim \Omega^2$ and $MR^2 \sim \mu a^2$, so

$$h \sim (G^2/c^4)(\mu/r)(M/a)\ . \tag{4}$$

This can also be written in terms of orbital periods, and with the correct factors put in we get, for example, for an equal mass system

$$h \approx 10^{-22}\left(\frac{M}{2.8\,M_\odot}\right)^{5/3}\left(\frac{0.01\text{ sec}}{P}\right)^{2/3}\left(\frac{100\text{ Mpc}}{r}\right)\ , \tag{5}$$

which is scaled to a double neutron star system. This is really, really, small: it corresponds to less than the radius of an atomic nucleus over a baseline the size of the Earth. That's why it is so challenging to detect these systems!

Remarkably, though, the flux of energy is *not* tiny. To see this, let's calculate the flux given some dimensionless amplitude h. The flux has to be proportional to the square of the amplitude and also the square of the frequency f: $F \sim h^2 f^2$. This currently has units of frequency squared, but the physical units of flux are energy per time per area. Replacing factors of G and c, we find that the flux is

$$F \sim (c^3/G)h^2 f^2\ . \tag{6}$$

Now the prefactor is *enormous*! For the double neutron star system above, with $h \sim 10^{-22}$ and $f \sim 100$ Hz, this gives a flux of a few hundredths of an erg cm^{-2} s^{-1}. For comparison, the flux from Sirius, the brightest star in the night sky, is about 10^{-4} erg cm^{-2} s^{-1}! That means that if you could somehow absorb gravitational radiation perfectly with your eyes, you would see hundreds to thousands of events per year brighter than every star except the Sun. To put it another way, the energy per time emitted by the GW150914 event, during the last part of its coalescence, was tens of times greater than the energy per time emitted by every star in the visible universe *combined* during that same time (!!!). What this really implies, of course, is that gravitational radiation interacts *very* weakly with matter, which again means that it is mighty challenging to detect.

Let us conclude with an idea of the frequency range available for a given type of binary. There is obviously no practical lower frequency limit (just increase the semimajor axis as much as you want), but there is a strict upper limit. The two objects in the binary clearly won't produce a signal higher than the frequency at which they touch. If we consider an object of mass M and radius R, the orbital frequency at its surface is $\sim \sqrt{GM/R^3}$. Noting that $M/R^3 \sim \rho$, we can say that the maximum frequency involving an object of density ρ is $f_{\rm max} \sim (G\rho)^{1/2}$. This is actually more general than just orbital frequencies. For example, a gravitationally bound object can't rotate faster than that, because it would fly apart. In addition, you can convince yourself that the frequency of a sound wave that involves most of the object can't be greater than $\sim (G\rho)^{1/2}$. Therefore, $\sim (G\rho)^{1/2}$ is a general upper bound on dynamical frequencies.

This tells us, therefore, that binaries involving main sequence stars can't have frequencies greater than $\sim 10^{-6} - 10^{-3}$ Hz, depending on mass, that binaries involving white dwarfs can't have frequencies greater than $\sim 0.1 - 1$ Hz, also depending on mass, that for neutron stars the upper limit is $\sim 1000 - 2000$ Hz, and that for black holes the limit depends inversely on mass (and also depends on the spin and orientation of the binary). In particular, for black holes the maximum imaginable frequency is few $\times\, 10^4 (M_\odot/M)$ Hz at the event horizon, but in reality the orbit becomes unstable at lower frequencies.

The net result is that for ground-based interferometers such as the two US-based LIGO detectors, GEO-600, Virgo, KAGRA, and LIGO-India, which are sensitive to frequencies $\sim 10 - 2000$ Hz, the only individual sources that will be detected are neutron stars and black holes and their creation events (supernovae); some might argue that cuspy cosmic strings might fall into this category, but there's no evidence for such things.

For a rigorous derivation of the evolution of a well-separated binary under the influence of gravitational radiation, see Peters 1964, Physical Review, 136, B1224. This classic paper derives the rates of energy and angular momentum loss, and hence the rate of change of the semimajor axis and angular momentum, for a binary which is imagined to move in a Keplerian way over a full orbit. This paper also does a good job of providing the framework in which other gravitational radiation calculations can be performed (e.g., of a lump on a rotating star).