Units, limits, and symmetries

When solving physics problems it's easy to get overwhelmed by the complexity of some of the concepts and equations. It's important to have ways to navigate through these complexities and reduce errors. One of the best navigation tools is a sense of what the answer should look like. What units should it have? How should it behave in easily-understood limits? What are the symmetries of the problem? What should the answer depend on? You should check every answer you get against these common-sense guides. This will cut down dramatically on errors in derivation. Even more importantly, it will help build up your intuition about physics, because you will be able to approach problems by constraining the answer first. It will also put you one step ahead of quite a number of researchers; you'd be amazed how often you'll be able to get to an answer quickly using these techniques!

In this class we'll run into a *lot* of equations, some of them pretty hairy. I want you to form the habit of checking each equation for units, limits, and symmetries (as well as other things such as conservation laws). To help with this, Doug Hamilton and I are writing a book of practice problems. In a given problem, you're presented with a physical situation and several possible expressions for the correct answer. The point it *not* to get the right answer per se, but to be able to rule out definitely wrong answers using simple arguments. In our class, I'll frequently stop and consider an equation to see if it can be the right one.

For this lecture, then, I want to give examples of this and practice on a few cases in mechanics, since these tend to be cleanest. To do this I am taking examples from the book Doug and I are writing, as well as some of the introductory material.

1. CHECKING UNITS

Units are the first thing to check when considering possible answers to a problem. Any equation that you write must be dimensionally correct. Check your equations occasionally as you go through a derivation. It takes just a second to do so, and you can quickly catch many common errors. Remember this general rule: in all physically valid solutions, the argument of all functions (e.g. trigonometric functions, exponentials, logs, hyperbolic functions, etc.) must be dimensionless. Taking the cosine of something with units of mass or length makes no physical sense. For each of the problems below, imagine that you and several friends have just gone through lengthy derivations and each come up with different answers. In each case, you can rule out several of the answers with dimensional arguments without ever having to look at the actual derivations.

Here's an example:

1. A daredevil is shot out of a cannon at speed v and angle θ from horizontal. Earth's gravitational acceleration, g, is assumed constant, and air resistance is neglected. How far downrange, D, does the daredevil fly before hitting the ground?

A) $D = 2v^2 \cos\theta$
B) $D = (2v^2/g)\sin\theta$
C) $D = 2g\sin\theta\cos\theta$
D) $D = 2vg(\cos\theta - \sin\theta)$
E) $D = (2v^2/g)\sin g$

Answer: Distance is measured in meters, velocities in m/s, and acceleration in m/s^2. All of the above answers have left-hand sides which are distances in units of meters, so the correct answer must have units of meters on the right as well.
A) has units of velocity squared (WRONG)
B) has units of meters (COULD BE OK)
C) has units of acceleration (WRONG)
D) has units of meter squared per second squared (WRONG)
E) the argument of the sine has units (WRONG)
Note that units checking like this is important but does have limitations. For example, any equation that is dimensionally correct is also dimensionally correct if either side is multiplied by an arbitrary dimensionless factor.

2. CHECKING LIMITS

Check all of your final answers and important intermediate results to see if they behave correctly in as many different limits as you can think of. Sometimes you will know how a general expression should behave if a particular variable is set to zero, infinity, or some other value. Make sure that your general expression actually displays the expected behavior!

Here is a simple example that can be used as a mnemonic for some trigonometric multiple angle formulae.

2. The double angle formula for sines is $\sin(2\theta) = 2\sin\theta\cos\theta$. Which of the following expressions might be correct generalizations?

A) $\sin(\theta_1 + \theta_2) = \sin\theta_1\cos\theta_2 - \cos\theta_1\sin\theta_2$
B) $\sin(\theta_1 + \theta_2) = \sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2$
C) $\sin(\theta_1 + \theta_2) = \cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2$
D) $\sin(\theta_1 + \theta_2) = \cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2$

Answer: We know how the correct equation must behave in a number of limits. If $\theta_1 = \theta_2$, for example, the correct expression must reduce to the formula for $\sin(2\theta)$ given above.
A) reduces to $\sin(2\theta) = 0$ (WRONG)
B) reduces to $\sin(2\theta) = 2\sin\theta\cos\theta$ (COULD BE OK)
C) reduces to $\sin(2\theta) = 1$ (WRONG)
D) reduces to $\sin(2\theta) = \cos^2\theta - \sin^2\theta$ (WRONG)
Note that the equation is wrong if it fails for *any* value of θ - so C) and D) are wrong because they fail for $\theta = 0°$ while A) is wrong because it fails for $\theta = 45°$. Note that you can obtain the formula for the sine of the difference of two angles by letting $\theta_2 \rightarrow -\theta_2$ in B).

3. TAKING ADVANTAGE OF SYMMETRIES

Symmetries are fundamental in physics (and astronomy!). Problems can have symmetry about a point (spherical symmetry), a line (cylindrical or axial symmetry), or a plane (mirror symmetry). You can use symmetries in two ways: 1) to check your final answer to a problem or, with a little more effort, 2) to simplify the derivation of that final answer. As an example, time-independent central forces (like gravity) have spherical symmetry because the force depends only on the distance from the origin. In this case, spherical symmetry means that once we find one solution (e.g. a particular ellipse for gravity), all other possible orientations of this solution in space are also solutions.

Another type of symmetry could be called a symmetry of labeling. In many problems, it is clear that simply renaming two identical things can't change anything fundamental about the system. For example, consider two objects of mass m_1 and m_2 moving in circular orbits around each other, bound by gravity, separated by a distance a. What is the frequency of rotation? A guess like $\omega = \sqrt{G(2m_1 + m_2)/a^3}$ can't be right, because the answer would change simply by switching the labels on the masses.

4. PUTTING IT TOGETHER

Now let's use these for some more problems in mechanics. For these, have class rule out one at a time (that is, ask for someone to rule out one answer; then another student rules out another answer; and so on).

3. Let a mass m be in a circular orbit of radius r and angular frequency ω radians per second. What is the centripetal force needed to keep the mass in that orbit?

A) $f = m$
B) $f = r\omega$

C) $f = mr\omega$
D) $f = mr\omega^2$
E) $f = mr^2\omega^2$

4. A good estimate for the energy released during the accretion of Jupiter is:

A) GM^2/R
B) GM^2/R^2
C) GM/R
D) GM/R^2

5. The sound speed c_s in an ideal gas depends only on the pressure P and the density ρ of the gas. The correct expression is:

A) $c_s = \sqrt{\rho/P}$
B) $c_s = \sqrt{P/\rho}$
C) $c_s = P^2/\rho$
D) $c_s = \rho^2/P$
E) $c_s = \sqrt{P\rho}$

6. Two bodies of masses m_1 and m_2 placed a distance r apart. What is the strength of the gravitational forces that the bodies exert on each other?

A) $F = G(m_1 + m_2)m_2/r^2$
B) $F = G(m_1 + r)(m_2 + r)/r^2$
C) $F = Gr^2 m_1 m_2$
D) $F = G/(r^2 m_1 m_2)$
E) $F = Gm_1 m_2/r^2$

7. Let a particle orbit in a circle a distance h above a planet of mass M and radius R (the particle mass is assumed very small). What is the angular momentum per unit mass of the particle?

A) $\ell = \sqrt{GM(R + h)}$
B) $\ell = \sqrt{GM}(Rh)^{1/4}$
C) $\ell = \sqrt{GMR^2/h}$

D) $\ell = \sqrt{GMh^2/R}$

8. A telescope with aperture D observes a source at a wavelength λ. Diffraction limits the angular resolution. What is that limit?

A) $\theta = 1.22\lambda D$.
B) $\theta = 1.22D/\lambda$.
C) $\theta = 1.22\lambda/D$.
D) $\theta = 1.22/(\lambda D)$.

9. A star of average radius R is rotating with angular frequency ω. We define the sign of ω such that if $\omega > 0$ then the star is rotating west to east like the Earth, whereas if $\omega < 0$ then the star is rotating east to west. Rotation will distort the radius of the star. To lowest order in ω, what will be the deviation of the equatorial radius from R?

A) $\Delta r \propto \omega$
B) $\Delta r \propto \omega^2$
C) $\Delta r \propto \omega^3$

10. A white dwarf of mass M has reached an equilibrium radius R. Its total energy is therefore a minimum. If the white dwarf's radius is changed by ΔR ($\Delta R < 0$ means shrinking the star; $\Delta R > 0$ means expanding it) then which of the following could be true about the change ΔE in the total energy?

A) $\Delta E \propto \Delta R$.
B) $\Delta E \propto (\Delta R)^2$.
C) $\Delta E \propto (\Delta R)^3$.
D) $\Delta E \propto (\Delta R)^4$.

11. You launch a rocket straight up from the Earth's North pole, and it rises up to a maximum height H, then falls back to Earth. The maximum height above the Earth is given by one of the expressions below. Here R_E is the Earth's radius, $X = v^2 R_E/GM_E$, G is the gravitational constant, M_E is the Earth's mass and v is the launch velocity. Rule out as many of the following expressions that you can.

A) $H = R_E X/(1 + \sqrt{X})$
B) $H = R_E X/(1 - X)$

C) $H = R_E X/(2 - X)$
D) $H = R_E(1 - X)/(2 - X)$
E) $H = vX^2/(2 - X)$
F) $H = R_E X/2$
G) $H = R_E X^2/(2 - X)$
H) $H = R_E X|1 - X|/(2 - X)$