Laws of Motion and Conservation Laws

The first astrophysics we'll consider will be gravity, which we'll address in the next class. First, though, we need to set the stage by talking about some of the basic laws of motion and the conservation laws that come from them. For this, I'll be drawing heavily on the *Feynman Lectures On Physics*, in addition to notes drawn up by Lee Mundy and Eve Ostriker.

Let's start with Newton's laws. **Ask class:** what is Newton's first law of motion? You'll often see it stated something like "an object remains in motion unless acted upon by an external force". This says that the velocity is constant in the absence of external forces: $\mathbf{v}$ =const. Note that this is a statement about the *vector* velocity; it is possible for the *speed* to remain constant even if the velocity changes. **Ask class:** can they think of an example of this? Constant speed motion in a circle, for example. Now, since in Newtonian physics the mass is constant, we can equally well write $m\mathbf{v}$ =const, or $\mathbf{p}$=const in the absence of external forces, where $\mathbf{p} = m\mathbf{v}$ is the linear momentum. This turns out to be the most general form of the law, valid in relativistic mechanics as well:

$$\text{If } \mathbf{F} = 0,\ d\mathbf{p}/dt = 0\ . \tag{1}$$

Ask class: what is Newton's second law of motion? This one you'll commonly see as $F = ma$, but since it's a vector equation we should really write $\mathbf{F} = m\mathbf{a}$, where $\mathbf{F}$ is the external force, m is the mass of the object, and $\mathbf{a}$ is the acceleration of the object. Since $\mathbf{a} = d\mathbf{v}/dt$ and m =const in Newtonian mechanics, $m\mathbf{a} = d\mathbf{p}/dt$, and we can write more generally

$$d\mathbf{p}/dt = \mathbf{F}\ . \tag{2}$$

In words, the time rate of change of the momentum of an object equals the external force on that object (where the equation is written as a vector equation). Again, this is correct relativistically as well as in Newtonian mechanics. Note that the second law includes the first, when written in this way. An auxiliary point is that the force $\mathbf{F}$ can be written as the vector sum of all the forces from every other object: $\mathbf{F} = \sum_i \mathbf{F}_i$, where in principle the sum runs over every particle in the universe! In practice, you can almost always simplify to a few nearby bodies or particles.

Ask class: what is Newton's third law of motion? You'll often see it as "for every action, there is an equal and opposite reaction". More specifically, consider two objects A and B. Let the (vector!) force of A on B be $\mathbf{F}_{AB}$, and the vector force of B on A be $\mathbf{F}_{BA}$. Then Newton's third law says

$$\mathbf{F}_{AB} = -\mathbf{F}_{BA}\ . \tag{3}$$

Note the amazing generality of this law. It doesn't matter *what* force you consider. It could be gravity, it could be electromagnetism, it could be the strong or weak force, it could be some oddball force no one has ever discovered before. Nonetheless, this symmetry applies. This imposes *tremendous* restrictions on the form a force may take. Another important restriction is that the force between two particles acts along the line between them. Thus, for example, if particle A is at vector location $\mathbf{r}_A$ in some coordinate system, and particle B is at vector location $\mathbf{r}_B$, then the force between them is $\mathbf{F}_{AB} \propto (\mathbf{r}_B - \mathbf{r}_A)$, where the proportionality constant can be positive or negative and depends on the nature of the force.

We now come to an important philosophical point. The two laws $d\mathbf{p}/dt = \mathbf{F}$ and $\mathbf{F}_{AB} = -\mathbf{F}_{BA}$ provide an unerring way to determine the motion of any classical system (quantum principles impose different guidelines for microscopic systems, but for macroscopic systems the laws reduce to the classical ones). If you are given a functional form for the force (e.g., how it depends on the distance between objects, or their mass or charge or whatever), and the initial positions and velocities of all objects, then in principle you can figure out the positions and velocities of all objects for any time in the future.

In practice, however, this is often impossibly complicated. It is also the case that $\mathbf{F} = m\mathbf{a}$, although completely general, may not be the best way to approach a particular problem. This is one reason why through the centuries people have developed alternative ways to do mechanics problems: Hamiltonian and Lagrangian are two examples. The point to remember is that although these points of view are entirely equivalent, looking at the problem a different way very often yields new insights. An especially important new perspective on the motion of systems comes from looking at overall quantities that are conserved, which is our next task.

For our investigation of conservation principles, let us postulate a system that is completely isolated from the rest of the universe. This simplifies things, and is often surprisingly accurate. For example, we can consider the motion of planets in the Solar System without worrying about the effect of the rest of the Milky Way galaxy. As a first step, in fact, we'll think just of two point particles exerting forces on each other. This approach is very valuable when constructing models for astrophysical phenomena: do it simply first, then generalize.

First, let's consider momentum. We'll name the two particles A and B as before. Let their momenta be $\mathbf{p}_A$ and $\mathbf{p}_B$. How do these momenta change with time? Using the same notation we had before,

$$\begin{aligned} d\mathbf{p}_A/dt &= \mathbf{F}_{BA} \\ d\mathbf{p}_B/dt &= \mathbf{F}_{AB}\ . \end{aligned} \tag{4}$$

But what does this mean about the total momentum $\mathbf{p}_{\rm tot} \equiv \mathbf{p}_A + \mathbf{p}_B$?

$$d\mathbf{p}_{\rm tot}/dt = d\left(\mathbf{p}_A + \mathbf{p}_B\right)/dt = \mathbf{F}_{BA} + \mathbf{F}_{AB} = \mathbf{F}_{BA} - \mathbf{F}_{BA} = 0\ . \tag{5}$$

Therefore, the total momentum is conserved. **Ask class:** what if there are more than two particles? Then, since the total force is the linear vector sum of each of the individual forces, one can repeat the procedure above for every pair of particles. The net result is that, again, the total linear momentum is conserved. This is a major result. It doesn't tell us how a system will move or evolve (we need Newton's laws and a force law for that), but it places serious constraints on the states an isolated system can attain.

Let's try another one. Select an arbitrary point in space, and measure vector positions $\mathbf{r}$ relative to that point. Select a reference frame in which that point is stationary, and in that reference frame determine the momenta $\mathbf{p}$ of particles. We then define the angular momentum $\mathbf{L}$ as

$$\mathbf{L} \equiv \mathbf{r} \times \mathbf{p} \; . \tag{6}$$

The time rate of change of the angular momentum is the torque $\mathbf{N}$:

$$\mathbf{N} \equiv d\mathbf{L}/dt = (d\mathbf{r}/dt) \times \mathbf{p} + \mathbf{r} \times (d\mathbf{p}/dt) = \mathbf{v} \times \mathbf{p} + \mathbf{r} \times \mathbf{F} = \mathbf{v} \times m\mathbf{v} + \mathbf{r} \times \mathbf{F} = \mathbf{r} \times \mathbf{F} \; . \tag{7}$$

How does the angular momentum evolve for an isolated system? Again, let's simplify to a two-particle system. Then the time rate of change of the total angular momentum $\mathbf{L}_{\rm tot} \equiv \mathbf{L}_A + \mathbf{L}_B$ is

$$\begin{aligned} d\mathbf{L}_{\rm tot}/dt &= \mathbf{r}_A \times \mathbf{F}_{BA} + \mathbf{r}_B \times \mathbf{F}_{AB} \\ &= -\mathbf{r}_A \times \mathbf{F}_{AB} + \mathbf{r}_B \times \mathbf{F}_{AB} \\ &= (\mathbf{r}_B - \mathbf{r}_A) \times \mathbf{F}_{AB} \\ &\propto (\mathbf{r}_B - \mathbf{r}_A) \times (\mathbf{r}_B - \mathbf{r}_A) \\ &= 0 \; . \end{aligned} \tag{8}$$

Therefore, for an isolated two-particle system, the angular momentum is conserved. **Ask class:** what happens for an isolated system with more than two particles? As with linear momentum, the angular momentum is still constant, as can be seen from pairwise cancellation. Thus, *for any isolated system, the linear momentum and angular momentum are constant*. Another way of putting this is that for *any* system, isolated or otherwise, the total linear and angular momentum are changed only by *external* forces and torques, respectively:

$$d\mathbf{p}_{\rm tot}/dt = \mathbf{F}_{\rm ext}, \qquad d\mathbf{L}_{\rm tot}/dt = \mathbf{N}_{\rm ext} \; . \tag{9}$$

You do have to be careful when applying these principles. Remember, for an isolated system it's the *total* linear and angular momenta that are constant. The linear or angular momentum of an individual component may or may not be constant. For example, consider the Earth-Moon system (assume for the moment it's isolated). We can break down the angular momentum into the sum of the Earth's spin angular momentum, the Moon's spin angular momentum, and the orbital angular momentum of the system. Over a long time, none of these three individual components are constant. In fact, the Moon is moving away from the Earth over billions of years. This increases the orbital angular momentum, and the spin angular momenta of the Earth and Moon decrease to compensate. Over even longer time scales, the torque exerted by the Sun takes angular momentum away from the Earth-Moon system.

Stated this way, it may seem that we're in real trouble, because we'll have to include the forces and torques from everything in the universe to conserve linear and angular momentum. It's not really that bad, though. If you want to be ultracautious about it, you can calculate the order of magnitude of the forces and torques outside the Solar System, and conclude that their effect is so tiny compared to forces and torques inside the Solar System that to any reasonable accuracy they can be neglected. Indeed, this is a procedure often used in astrophysics. To get an idea about some effect or phenomenon, you pare down the phenomena you include to only those that are strongest. After understanding the physics at that level, you may include weaker forces in order to look for more subtle effects. Simplify first, though!

We now move on to a third conservation principle: the conservation of energy. Unlike linear and angular momentum, energy is tricky to define rigorously. One try might be "Energy is a quantity which may be converted into motion". If all you have to worry about is energy of motion, it's straightforward: $E = \frac{1}{2}mv^2$ for nonrelativistic motion. However, there are many other forms of energy. Potential energy, electrostatic energy, thermal energy, chemical energy, nuclear energy, energy in photons or neutrinos or gravitational waves, and so on. Not all of those forms are distinct at a fundamental level, but it doesn't matter. In a general case, it can be difficult to track where all the energy goes, but the principle of energy conservation is that for an isolated system (as always), the total energy in all forms is constant, although the amount of energy in each form can change.

Feynman used a nice analogy for this, which I'll paraphrase closely. Suppose a child receives a toy for Christmas: 28 indestructible blocks. He plays with them in his room. He's rather messy, so his father comes in to clean up every once in a while. After a while, the father notices an amazing thing: day after day, there are always 28 blocks in the room! One day, there are only 27 visible; a search reveals that one of the blocks is under the rug, though, so the total is still 28. Another day there are only 26. Careful examination shows that the window is open, and indeed the two missing blocks are outside. Another day there are 30 blocks! Yikes! But it turns out that the child had a friend visiting, and the friend brought two blocks with them, so that's okay. The next day brought a puzzle; only 25 blocks are visible, and the father has searched everywhere but the toy box, which is closed. The child throws a tantrum and refuses to allow the box to be opened. Being clever, however, the father weighs the toy box and discovers that it has excess weight exactly equal to three blocks. On yet another day, only 20 blocks can be seen. After all the other possibilities have been eliminated, the father notes that the bath water is higher than it was when the water was poured in. The bath water is dirty, so the father can't directly check if the blocks are in there, but using Archimedes' principle he finds that just the right amount of water is displaced for 8 blocks. As time goes on, in fact, he discovers that there are always 28 blocks, although creativity may be required to discover where they are.

One reason that we're going to focus first on pure gravity is that with a limited number of interactions it's easier to find where the energy goes. When we discuss gas physics, it becomes tougher. However, like linear and angular momentum conservation, energy conservation is a fundamental law of nature that gives us overall constraints on the possible states of a system. You will simplify your life enormously if you pay attention to conservation laws. Here's one example from mechanics. Suppose you have an object in a constant gravitational field g, a height h above a table. If you drop it from rest, how fast is the object moving when it hits the table?

First let's do it with old $F = ma$, where we're only considering one direction (down), so we don't write this explicitly as a vector equation. The acceleration is g. Since the object starts at rest, we have $d^2z/dt^2 = -g$, with the boundary conditions $z(0) = h$, $\dot{z}(0) = 0$, where the dot means a time derivative. Solving, $z = h - \frac{1}{2}gt^2$. This means that $z = 0$ (i.e., the object hits the table) at a time $t = \sqrt{2h/g}$. We also know that the speed as a function of time is $v = \dot{z} = -gt$, so when the object hits the table, $v = -g\sqrt{2h/g} = -\sqrt{2gh}$.

Now let's do it using conservation of energy. The potential energy initially is mgh. The kinetic energy when it hits is $\frac{1}{2}mv^2$. Equating the two, $v = -\sqrt{2gh}$, where the negative sign is because it's going downwards. Much easier!

I'll conclude with some final comments about conservation laws. From the quantum mechanical standpoint, the laws we've discussed come ultimately from symmetries about the universe. The conservation of energy exists because whether you perform an experiment in an isolated laboratory today or tomorrow you'll get the same answer; this is symmetry in time. Linear momentum is conserved because whether you perform an experiment in an isolated laboratory here or a megaparsec from here, you'll get the same answer. Angular momentum conservation occurs because you can rotate your isolated laboratory (not set it spinning, but rotate it through some angle and then stop) without affecting the outcomes of experiments. In addition to these conservation principles, there are others obtained from particle physics that are in a sense easier to understand because, as Feynman says, they are more like counting blocks. The total electrical charge is a constant, as is the total number of leptons and of baryons (two types of particles), as long as you count an antiparticle as -1 of a particle. Together, these conservation laws form the foundation of much of modern physics.