One-body gravitation, continued

Let's continue by investigating the period of an orbit under the influence of an inverse square force. Our next question has to do with the period of the orbit. When doing a derivation such as this, it's useful to start by establishing a known path to the solution, even if that path is nonoptimal. In our case, we know that

$$\begin{aligned} \dot{\theta}r^2 &= j \\ (d\theta/dt)r^2 &= j \\ (r^2/j)d\theta &= dt\ . \end{aligned} \tag{1}$$

The period is $P = \int dt$, where the integral is over a full period $\theta = 0$ to 2π. Since we know r as a function of θ, we can do this integral and get the answer. Perhaps the integral will require use of an integral table, but the answer can be obtained in this way.

Now that we know we can solve it that way, we can step back and examine whether anything about the problem allows a trick to get to the solution more elegantly. In this case there is such a trick, and it stems from Kepler's second law of equal areas swept out in equal times. Consider the area swept out by a particle initially at position $\mathbf{r}$ that moves to a position $\mathbf{r} + \mathbf{\Delta r}$. The area of this triangle is

$$\Delta A = \frac{1}{2}|\mathbf{r} \times \mathbf{\Delta r}|\ , \tag{2}$$

so the constant rate at which the area is swept out is

$$dA/dt = \frac{1}{2}|\mathbf{r} \times \mathbf{v}| = \frac{1}{2m}|\mathbf{r} \times m\mathbf{v}| = L/(2m) = j/2\ . \tag{3}$$

Therefore, the time needed to sweep out an area A_{12} between points 1 and 2 is $t_{12} = A_{12}/\dot{A} = A_{12}(2/j)$. The area of an ellipse of major axis a and minor axis b is $A = \pi ab$, so the period of the orbit is

$$P = 2\pi ab/j = \frac{2\pi a^2}{j}\sqrt{1-e^2}\ , \tag{4}$$

where the last equality is because for an ellipse $b/a = \sqrt{1-e^2}$. We now turn to the polar equation for the orbit,

$$r = r_0\frac{1+e}{1+e\cos\theta}\ , \tag{5}$$

to realize that at $\theta = 0$, $r = r_0 = j^2/[GM(1+e)]$, and at $\theta = \pi$, $r = r_1 \equiv r_0(1+e)/(1-e) = j^2/[GM(1-e)]$. Therefore, the major axis is

$$2a = r_0 + r_1 = \frac{j^2}{GM}\left(\frac{1}{1+e} + \frac{1}{1-e}\right) = \frac{2j^2}{GM(1-e^2)}\ . \tag{6}$$

Writing $(1-e^2)$ in terms of a allows us to eliminate $(1-e^2)$ in the expression for the period, and after some rearranging we get

$$P = 2\pi \left(a^3/GM\right)^{1/2} \ . \tag{7}$$

For a system with a dominant mass, such as the solar system, GM is the same for all planets and we get *Kepler's third law: the square of the orbital period is proportional to the cube of the semimajor axis.*

Again, we could have obtained exactly the same answer by a direct integration. Which one you use is a matter of personal preference. It is, however, useful to look at alternate ways of doing such calculations so that you have a larger bag of tricks.

One thing that's been missing from our discussion so far is energy. When you drop a stone in a gravitational field, it speeds up as it falls. Conservation of total energy requires that some form of energy decrease to compensate for the increase in kinetic energy. This form of energy is *potential energy.* Potential energy is only useful as a concept if the force in question is *conservative.* What does that mean? One operational definition is that a force is conservative if a particle can be moved in an arbitrary path, returning to its starting point, and have the same energy it did initially. The work done by a force $\mathbf{F}$ over an infinitesimal path $d\mathbf{s}$ is $\mathbf{F}\cdot d\mathbf{s}$, so this condition is

$$\oint \mathbf{F}\cdot d\mathbf{s} = 0 \tag{8}$$

for any closed path. **Ask class:** can they think of an example of a force that violates this condition? Any drag force will do it, since drag forces are always opposite to the direction of motion. If the condition above does hold, it also means that the work W_{12} done by the force between any two points 1 and 2 doesn't depend on the path taken between those points. This means it is possible to express W_{12} as the change in something that depends only on where those two points are. That something must therefore be a scalar, which we'll call $-V$. Differentially, then, we have

$$\begin{aligned} \mathbf{F}\cdot d\mathbf{s} &= -dV \\ F_s &= -\partial V/\partial s \\ \mathbf{F} &= -\nabla V(\mathbf{r}) \ . \end{aligned} \tag{9}$$

This is an intuitive derivation of something that can be proved rigorously in calculus: *for a force to be conservative, it is necessary and sufficient that it be expressible as the gradient of a scalar function of position only.* This V is called the potential energy. Like the force, the potential energies from a collection of particles add linearly. Notice that we can add any constant to V without changing the force, hence the potential energy does not have an overall normalization. Conversely, this means we can select a normalization that is convenient. The usual one is that the potential energy goes to zero as $r \to \infty$. To return to our orbit problem, this means that if and only if the force field is conservative, the total energy of a particle (V

plus the kinetic energy $\frac{1}{2}mv^2$) remains constant. Note, in particular, that if the potential varies with time in the region that the particle traverses, the energy of the particle need not remain constant. Of course, as we emphasized in our discussion of conservation laws, the *total* energy of the system must be conserved, but for now we're focusing on the energy of just one particle.

How can we tell if a given force can be expressed as a gradient of a scalar? Recall that $\nabla \times \nabla A = 0$, where A is any scalar. It goes the other way, too: if $\nabla \times \mathbf{B} = 0$, where $\mathbf{B}$ is some vector, then $\mathbf{B} = \nabla A$ can be expressed as a gradient of a scalar. Therefore, if for a force $\mathbf{F}$, $\nabla \times \mathbf{F} = 0$, then $\mathbf{F}$ is conservative. As a particular example, any central force $\mathbf{F} = f(r)\hat{\mathbf{r}}$ is conservative. Specifically, therefore, gravity is conservative. Since $\mathbf{F}_{\rm grav} = -(GM/r^2)\hat{\mathbf{r}} = -\nabla V_{\rm grav}(r)$, the gravitational potential energy is

$$V_{\rm grav}(r) = -\frac{GMm}{r} \; . \tag{10}$$

Let's consider a couple of examples. Consider a thin spherical shell of radius R and total mass M. Normalize V so that $V = 0$ at infinity. **Ask class:** what is the potential of a small mass m a distance $r > R$ from the center of the shell? It's $-GMm/r$. **Ask class:** what is the potential of a small mass m inside the shell, at a distance $r < R$ from the center? It's a constant, $-GMm/R$. Note that the gradient of a constant is zero, so the force is zero inside the shell as we derived earlier. Note that, if we define $V = 0$ at infinity, then because $V < 0$ closer to an object, it requires an input of energy to take a small mass from near an object out to infinity; the small mass is gravitationally bound to an object. The magnitude of V is therefore the energy input required to unbind a small mass, or equivalently, is the amount of kinetic energy generated by a small mass falling from a great distance.

As another example, consider the following. Protostellar evolution effectively involves a gas cloud gradually becoming more and more compact as it settles down to become a star. **Ask class:** to within an order of magnitude, what is the total energy liberated in the production of a star of mass M and radius R? To this accuracy, you can imagine each little bit of matter m falling onto a mass M. The energy liberated is GMm/R. But to assemble the star, we need not a mass m, but a mass M. Therefore, to order of magnitude, the total energy release is GM^2/R. If we assume that the luminosity during this phase is roughly equal to the luminosity of the star on the main sequence, we can get an estimate for the duration of the protostellar phase. For the Sun, that number is about 3×10^7 yr.

Armed with our potential energy, we can define a total energy E that is the sum of the kinetic and potential energies. In polar coordinates,

$$E = \frac{1}{2}m(\dot{r}^2 + r^2\dot{\theta}^2) + V(r) = \text{constant.} \tag{11}$$

Remarkably, for inverse square orbits, the value of the semimajor axis a is determined entirely from the total energy:

$$a = -GMm/(2|E|) \ . \tag{12}$$

In addition, the eccentricity can be written as a function of E and the other parameters:

$$e = \left[1 + 2Emj^2/(GMm)^2\right]^{1/2} \ . \tag{13}$$

From this, we can see that the type of orbit depends on the energy:

$$\begin{array}{ll} E < 0 & e < 1 : \text{closed orbits (circle or ellipse)} \\ E = 0 & e = 1 : \text{parabolic orbit} \\ E > 0 & e > 1 : \text{hyperbolic orbit} \ . \end{array} \tag{14}$$

We're really rather lucky mathematically that gravity is an inverse square force. Orbits are closed (they retrace themselves) and are stable (i.e., a slight perturbation of an elliptical orbit gives a slightly different elliptical orbit, but nothing catastrophic happens). This isn't true in general for a central force. For example, consider $F \propto r^n$. Bertrand's Theorem is that the *only* force laws of this type that give closed orbits are $n = -2$ (our inverse square) and $n = 1$ (Hooke's law, for a spring)! The lack of closure in other cases leads to *precession.* That is, for example, suppose you determine the angle $\theta_{\rm peri}$ of closest approach (pericenter) for an orbit. If $n = -2$ or $n = 1$, then $\theta_{\rm peri}$ is the same, orbit after orbit. If the force law is something else, $\theta_{\rm peri}$ changes. This is indeed the case for general relativistic gravity, which effectively is steeper than r^{-2} near an object. The precession induced by this effect is seen in an anomalous extra 43" per century of precession by Mercury (the total precession is about 100 times this, but is the result of perturbations by other planets).

Stability is another issue. **Ask class:** I assert that a circular orbit is possible in any central force field. Why? It's because $F = F(r)$ only, so at constant r one has a constant force. However, that orbit doesn't have to be stable. It turns out that if $n \leq -3$, circular orbits are *unstable*, meaning that a slight perturbation will cause the orbit to either spiral inwards or outwards. Once again, general relativity provides an example. Its effective force law isn't really a power law, but the force increases faster than $1/r^2$, and close enough to a really compact object (a black hole or neutron star), circular orbits are unstable. Matter that gets that close spirals in quickly. The existence of such unstable orbits is crucial to the understanding of compact objects and the astrophysical phenomena that arise from them.

Our last task today will be to consider some aspects of the potential from a *nonspherical* mass. This will give us some practice in perturbative expansions, and will be our first encounter with tidal forces.

As usual, we'll simplify to the max. Rather than thinking about an arbitrary nonspherical distribution, we'll consider the potential produced by two point masses. Suppose that the masses are m_0 and m_1, that they are separated by a distance R, and that we are interested in the potential energy of a particle of mass m_2 that is a distance r away from the center of mass. We also need an angle: we'll assume that the angle made from m_2, to the center of mass, to m_1, is ϕ.

The total potential of m_2 is

$$V = -Gm_0m_2/r_{02} - Gm_1m_2/r_{12} \ , \tag{15}$$

where r_{02} is the distance from m_0 to m_2 and r_{12} is the distance from m_1 to m_2. Let's define a_0 to be the distance from the center of mass to m_0 (so that $a_0 = Rm_1/(m_0+m_1)$), and $a_1 = Rm_0/(m_0+m_1)$ to be the distance from the center of mass to m_1. Then by the law of cosines,

$$r_{02}^2 = r^2 + a_0^2 + 2ra_0\cos\phi \ , \qquad r_{12}^2 = r^2 + a_1^2 - 2ra_1\cos\phi \ . \tag{16}$$

We'd like to do a perturbative expansion, that is, we want to know just the lowest order effect of this nonspherical distribution. We therefore assume that $r \gg R$, and expand in a_0/r and a_1/r. We therefore start by writing

$$r_{02}^2 = r^2\left[1 + 2(a_0/r)\cos\phi + (a_0/r)^2\right] \ , \qquad r_{12}^2 = r^2\left[1 - 2(a_1/r)\cos\phi + (a_1/r)^2\right] \ . \tag{17}$$

We need $1/r_{02}$ and $1/r_{12}$, so we need expansions of a square root and a reciprocal. Normally we'd just do this to first order, but as you'll see we actually need second order for this calculation. To second order, for a quantity $\epsilon \ll 1$, $\sqrt{1+\epsilon} \approx 1 + \epsilon/2 - \epsilon^2/8$ and $1/(1+\epsilon) \approx 1 - \epsilon + \epsilon^2$. Applying these, and doing a little simplifying, we get

$$\frac{1}{r_{02}} \approx \frac{1}{r}\left[1 - \frac{a_0}{r}\cos\phi + \frac{a_0^2}{2r^2}(3\cos^2\phi - 1)\right] \ , \qquad \frac{1}{r_{12}} \approx \frac{1}{r}\left[1 + \frac{a_1}{r}\cos\phi + \frac{a_1^2}{2r^2}(3\cos^2\phi - 1)\right] \ . \tag{18}$$

To get the total potential, it is most useful to separate out into powers of R/r, after we've written out the expressions for a_0 and a_1:

$$\begin{aligned} V \ \approx\ & -\tfrac{Gm_0m_2}{r} - \tfrac{Gm_1m_2}{r} + \tfrac{Gm_0m_2}{r}\tfrac{m_1}{m_0+m_1}\tfrac{R}{r}\cos\phi - \tfrac{Gm_1m_2}{r}\tfrac{m_0}{m_0+m_1}\tfrac{R}{r}\cos\phi \\ & -\tfrac{Gm_0m_2}{r}\tfrac{m_1^2}{(m_0+m_1)^2}\tfrac{R^2}{r^2}\tfrac{1}{2}(3\cos^2\phi - 1) - \tfrac{Gm_1m_2}{r}\tfrac{m_0^2}{(m_0+m_1)^2}\tfrac{R^2}{r^2}\tfrac{1}{2}(3\cos^2\phi - 1) \end{aligned} \tag{19}$$

Notice that the R/r terms cancel! If we now define $M \equiv m_0 + m_1$ (total mass) and $\mu \equiv m_0m_1/(m_0+m_1)$ (reduced mass),

$$V \approx -\frac{GMm_2}{r}\left[1 + \frac{\mu}{M}\frac{R^2}{r^2}\frac{1}{2}(3\cos^2\phi - 1)\right] \ . \tag{20}$$

As always, after doing a derivation we need to check it. First, are the units correct? The factor GMm_2/r out front has the right units, so we need to know if the quantity in brackets is dimensionless. The 1 certainly is. The other term is a product of two things that are clearly dimensionless (μ/M is, because both are masses, and R^2/r^2 is, because they are both squared lengths). The remaining factor is also clearly dimensionless. Therefore, the units are correct. What about symmetry? The potential obviously can't change if we simply decide to relabel the masses so that m_0 becomes m_1 and vice versa, so our expression needs to reflect that and it does. What about limits? If, say, $m_1 \ll m_0$ then the potential should reduce to that of m_0 by itself. Does it? In that case, $M \approx m_0$ and $\mu \ll M$, so the potential becomes $-Gm_0m_2/r$, as it should. When $r \to \infty$, the mass distribution looks like a point and therefore the potential should look like the potential from a point mass, and it does. These checks give us more confidence that we're on the right track.

The form of this potential tells us a number of things. First, it gives an idea of how much the potential (and therefore the force) deviates from the potential of the total mass, as determined by the distance from the center of mass. As R/r becomes smaller, the deviations become much smaller; when $R/r < 0.1$, it's less than 1%, for example. Second, a particle orbiting in this potential sees variation with ϕ. **Ask class:** what does this mean about the angular momentum? It means that the angular momentum of the particle's orbit is *not* constant. In fact, this is one aspect of tidal coupling. Angular momentum can be transferred from spin to orbit or vice versa. We'll encounter more about such potentials when we discuss three-body gravity.