Potential/density pairs and Gauss's law

We showed last time that the motion of a particle in a cluster will evolve gradually, on the relaxation time scale. This time, however, is much longer than the typical orbital time, by a factor of order $0.1N/\ln N$ where N is the number of particles. This suggests that individual orbits are not affected much by the granularity of the particle distribution, so that such orbits may be approximated by assuming that the rest of the mass is smoothed out. In this class we will consider what such individual orbits look like. We will also consider how the gravitational potential distribution relates to the density distribution, and use of Gauss's law. Among the applications for this work are the use of observed stellar orbits as a powerful probe of the density distribution in the centers of galaxies, and the realization that the central regions of galaxies can be fed gas and stars very efficiently by certain families of orbits.

Recall from earlier that just as the forces from individual particles add linearly, so do the potentials. That is, the acceleration of some particle i, with mass m_i, is given by

$$\ddot{\mathbf{r}}_i = \mathbf{F}_{\rm tot}/m_i = \sum_{j\neq i} \mathbf{F}_{ij}/m_i \tag{1}$$

and if we define V_{ji} as the potential at i due to particle j, and let $\Phi_{ji} = V_{ji}/m_i$, then with $\Phi_i = \sum_{j\neq i} \Phi_{ji}$ we have

$$\ddot{\mathbf{r}}_i = -\nabla\Phi_i\ . \tag{2}$$

Since we are considering a smoothed mass distribution, we can remove the subscripts, and say that the acceleration of a test particle at any location is equal to minus the gradient of the potential at that location:

$$\ddot{\mathbf{r}} = -\nabla\Phi\ . \tag{3}$$

So what happens to an orbit with some potential? To start us off, let's assume that the mass distribution is spherically symmetric and time-independent. **Ask class:** what kind of a force law do we have then? The force can only depend on the radius, and be in the radial direction, therefore this is a central force. As a result, for any spherically symmetric distribution of matter, we can immediately carry over results from central forces. For example, the energy and angular momentum of an individual particle are conserved. Suppose that at a given radius r we approximate the slope of the central force by a power law, r^{-n}. **Ask class:** what is the steepest that such a force can be, i.e., what is the maximum value of n? The force is steepest when all the mass is concentrated at the center, since otherwise the mass outside a given radius doesn't contribute. Therefore, the steepest force is one in which there is a single object at the center with all the mass, so $n \leq 2$! Real forces from extended mass distributions are less steep than that. This implies that orbits are stable (since $n \geq 3$ is required for unstable orbits) and that orbital precession of the pericenter, if any, is retrograde.

Now let's consider somewhat more general potentials. Suppose that Φ is time-independent. **Ask class:** what is an example of this? Any equilibrium distribution is time-independent. Note that here, as in many places in astrophysics, we're not making a mathematically rigorous statement about the time dependence. Over a long time comparable to the relaxation time, we know that the density distribution, and hence the potential, will change. However, we are only interested in the changes over periods comparable to an orbit, so for large N the potential is effectively constant in that time. With time-independence, we have

$$\frac{d}{dt}\left(\frac{1}{2}|\dot{\mathbf{r}}|^2\right) = \dot{\mathbf{r}} \cdot \ddot{\mathbf{r}} = -\dot{\mathbf{r}} \cdot \nabla\Phi = -\frac{dx}{dt}\frac{\partial\Phi}{\partial x} - \frac{dy}{dt}\frac{\partial\Phi}{\partial y} - \frac{dz}{dt}\frac{\partial\Phi}{\partial z} = -\frac{d\Phi}{dt}\Big|_{\mathbf{r}=\mathbf{r}(t)} . \tag{4}$$

Therefore, $\frac{d}{dt}(\frac{1}{2}|\dot{\mathbf{r}}|^2 + \Phi) = \frac{d}{dt}(E_{\rm tot}/m) = 0$, so the total energy of the particle is constant. This is a reflection of a fundamental symmetry: time-independence implies conservation of energy. If the potential Φ is not constant in time, then $dE/dt = \partial\Phi/\partial t$.

We can get some additional insight by writing out the equation of motion in cylindrical coordinates:

$$(\ddot{r} - r\dot{\phi}^2)\hat{r} + \frac{1}{r}\frac{d}{dt}(r^2\dot{\phi})\hat{\phi} + \ddot{z}\hat{z} = -\frac{\partial\Phi}{\partial r}\hat{r} - \frac{\partial\Phi}{\partial z}\hat{z} - \frac{1}{r}\frac{\partial\Phi}{\partial\phi}\hat{\phi} . \tag{5}$$

Suppose we have an axisymmetric potential, i.e., $\partial\Phi/\partial\phi = 0$. **Ask class:** what does that imply, from the equation of motion? The right hand side has a $\hat{\phi}$ component that vanishes, so the left hand side must as well, meaning that

$$\frac{d}{dt}(r^2\dot{\phi}) = 0 \Rightarrow r^2\dot{\phi} = j = \text{constant}. \tag{6}$$

Therefore, the angular momentum in the $\hat{z}$ direction is conserved. This is another fundamental symmetry: axisymmetry implies conservation of angular momentum.

Now we can return to the spherically symmetric case. In spherical symmetry, all planes through the center are equivalent. If one picks a particle with a particular velocity $\mathbf{v}$ and radius vector $\mathbf{r}$, then the $\mathbf{v} - \mathbf{r}$ plane goes through the center. We have freedom in defining our coordinates, so we can choose this plane to be the $z = 0$ plane. **Ask class:** what, then, is $\ddot{z}$? It's zero. There is up-down symmetry about the $z = 0$ plane, so it doesn't go one way or the other. Therefore, the orbit remains in the same plane forever. With this in mind, and because we still have $\partial\Phi/\partial\phi = 0$, the equation of motion reduces to the single component

$$\begin{aligned} \ddot{r} - r\dot{\phi}^2 &= -\partial\Phi/\partial r \\ \ddot{r} - j^2/r^3 &= -\partial\Phi/\partial r . \end{aligned} \tag{7}$$

But this is *exactly* the equation of motion for a central force law, $\mathbf{F} = F(r)\hat{r}$. That's as expected intuitively. Note that if the potential is merely axisymmetric then in general the orbital plane can change by precessing about the axis, but that if the orbit stays roughly in the plane $z \approx 0$ then the equation of motion is close to the one for spherical symmetry because the particle doesn't "feel" the change in z.

More complicated potentials will in general require integration of all the equations directly. As always, note that the total energy and angular momentum of the system is constant. However, for time-variable or nonaxisymmetric potentials, the energy or angular momentum of an individual particle does not have to be conserved. One of the reasons that violent relaxation is so effective at shuffling the energies of particles is that $\partial\Phi/\partial t \neq 0$ due to the dynamical evolution of the mass distribution. Therefore, significant changes in the energy are common.

We now move on to potential-density pairs. That is, given a particular distribution of matter, we'd like to know the potential as a function of position, or vice versa. An important tool here is Poisson's equation. This equation states that

$$\begin{aligned} \nabla^2\Phi &= 4\pi G\rho \\ \text{or } \nabla\cdot\mathbf{F} &= -4\pi G\rho\ . \end{aligned} \tag{8}$$

Divergences always relate to sources, so this equation is a quantitative statement of the principle that mass is the source of gravitational force. If one integrates this over a volume, one has

$$\int_V \nabla\cdot(\nabla\Phi)d^3V = \int_V 4\pi G\rho d^3V = 4\pi GM\ . \tag{9}$$

We can rewrite this using the divergence theorem. The divergence theorem states that the integral of the divergence of some vector $\mathbf{f}$ over a volume is equal to the integral of $\mathbf{f}\cdot\hat{n}$ over the surface of that volume, where $\hat{n}$ is a unit vector perpendicular to the surface at each point:

$$\int_V (\nabla\cdot\mathbf{f})d^3V = \int_S \mathbf{f}\cdot\hat{n}d^2S\ . \tag{10}$$

Applying this to the integral above gives *Gauss's theorem*:

$$\int_S (\nabla\Phi)\cdot\hat{n}d^2S = 4\pi GM\ . \tag{11}$$

Poisson's equation and Gauss's theorem allow us to get the potential from a given density distribution, or vice versa. Gauss's theorem also allows us to prove Newton's Second Theorem a whole lot more easily than we did before! If the mass distribution is spherically symmetric, $\nabla\Phi$ must be purely radial. Therefore, $\nabla\Phi$ is already normal to the surface, so the dot product with $\hat{n}$ gives unity. In addition, symmetry guarantees that $\nabla\Phi$ is constant at a constant radius. Then

$$\begin{aligned} \textstyle\int_S \nabla\Phi d^2S &= 4\pi GM \\ 4\pi R^2\nabla\Phi &= 4\pi GM \\ \nabla\Phi &= GM/R^2 \\ F/m = (-\nabla\Phi)\hat{r} &= -(GM/R^2)\hat{r}\ . \end{aligned} \tag{12}$$

Piece of cake!

There are a number of specific potentials that are used for stellar distribution modeling. We've already encountered one in which a single mass dominates (e.g., the Sun in the Solar System). Then ρ is concentrated in a very high density in the center. This can be written in terms of a *Dirac delta function*, usually written $\delta(\mathbf{r})$. In mathematical formalism, the delta function has the property that it is zero everywhere except for one point, but at that point is infinite, so that the integral over all space is exactly one:

$$\begin{aligned} \delta(x) &= 0, \qquad x \neq 0 \\ &= \infty, \qquad x = 0 \\ \textstyle\int_{-\infty}^{\infty} \delta(x)\, dx &= 1\ . \end{aligned} \tag{13}$$

In this example the delta function is one-dimensional, but it can have any number of dimensions. Note from the integral that the delta function has units of the reciprocal of its argument. For example, if x represents length, then dx has units of length and therefore for the integral to come up with a unitless 1, $\delta(x)$ must have units of inverse length. For a three-dimensional delta function, e.g., $\delta(\mathbf{r})$, we have $\int \delta(\mathbf{r}) d^3r = 1$, so the delta function has units of inverse length cubed. In any case, this means that if we approximate the Sun as a point, the density distribution is $\rho(\mathbf{r}) = M_\odot \delta(\mathbf{r})$. Of course the density doesn't *really* go to infinity, but this is a useful approximation. If you solve for Φ, you find that $\Phi = -GM_\odot/r$, as expected.

Such a density distribution is called singular, because the point $r = 0$ has a singularity (infinite density). Real density distributions aren't like that. An example of a potential that is not singular is Plummer's potential

$$\Phi(r) = -\frac{GM}{\sqrt{r^2 + b^2}}\ , \tag{14}$$

where M is the total mass and b is some constant, with units of distance. You can see that far from the center, $r \gg b$, this is approximately the potential of a point mass. However, close to the center, $r \ll b$, the potential approaches a constant and therefore the force goes to zero. The associated density is

$$\rho(r) = \frac{3M}{4\pi b^3}\left(1 + \frac{r^2}{b^2}\right)^{-5/2}\ . \tag{15}$$

Note that this goes to constant density in the center, but drops off rapidly (like r^{-5}) at large radii.

If the density distribution (and therefore the potential) are spherically symmetric, then the energy, angular momentum, and orbital plane are conserved for each particle. However, since in general the force law is not $1/r^2$, the orbits are not ellipses and therefore don't close on themselves. One is left with precession that is retrograde for $1/r^n$, where $n < 2$.

Therefore, for a given particle the pericenter (closest approach to the center) and apocenter (furthest distance from the center) are constant, but the angle is not. Over many orbits this means that the particle traces out a rosette-like pattern.

More generally, a density distribution need not have any particular symmetry. It will likely be in equilibrium, because otherwise it will undergo rapid evolution. Therefore, a given particle's energy but not its angular momentum are conserved. For example, for a density distribution that is triaxial, the particles can undergo "box orbits" that pass arbitrarily close to the center.

In general, to build a model of a galaxy you need to (1) find a collection of orbits that self-consistently yields the density distribution, and (2) make sure that the orbits in the potential of that density distribution are self-consistent. Therefore, the types of orbits are directly related to galaxy morphologies. For example, box orbits are particularly important in making the "bar" part of a barred spiral.