Hydrodynamics

Although most fluid in the universe is not dramatically out of *equilibrium* per se (in the sense that the fluid will change its properties dramatically in a dynamical time scale), there are many cases in which the fluid is not *static* (i.e., in which the fluid is moving). The study of moving fluids is hydrodynamics.

There are a small number of equations that form the basis of hydrodynamics, and that's what we'll focus on in this lecture. Before giving the equations, however, here is their physical meaning. Note that we are assuming nonrelativistic fluids, so matter and energy are conserved separately.

- The mass of a fluid element is conserved.
- The energy of a fluid element is conserved.
- Fluid elements move in response to forces on them: $\mathbf{a} = \mathbf{F}/m$. This is the conservation of momentum.

When we talk about "fluid elements", we are imagining that for short times and distances we can figuratively slice up the fluid into little chunks and examine their motion. The motion of the fluid as a whole is then composed of the motion of the chunks, in the same way that in calculus a curve is built up of infinitesimal segments.

Since we're thinking about the motion of individual fluid elements, we have to be a little careful when we think about their motion. If one talks about the velocity field of a fluid, it is the velocity at a given instant of every point in the fluid. That is, this is the velocity as related to some fixed coordinate system in space. However, we are following fluid elements, which move in space. Therefore, when we think about the velocity of fluid elements, we must include both the changes in the velocity due to forces at a given location and the changes in velocity due to the fact that in some time dt the fluid element has moved, so the velocity field changes with the change in location. Specifically, this means that the total derivative in time of the velocity (which is the one relevant for comparison with $\mathbf{F}/m$, for example) is

$$d\mathbf{v}/dt = \partial\mathbf{v}/\partial t + (\mathbf{v}\cdot\nabla)\mathbf{v}\ . \tag{1}$$

In Cartesian coordinates, this says

$$d\mathbf{v}/dt = \partial\mathbf{v}/\partial t + v_x(\partial\mathbf{v}/\partial x) + v_y(\partial\mathbf{v}/\partial y) + v_z(\partial\mathbf{v}/\partial z)\ . \tag{2}$$

This kind of change from a fixed frame to a moving one (or, an Eulerian description to a Lagrangian description) also enters when describing the change in other quantities, such as the entropy.

Now let's turn the three conservation principles into equations. I'm following Landau and Lifshitz, *Fluid Mechanics*, here.

Conservation of mass.—In order to show you how a derivation like this proceeds, we'll give details. Consider some volume V_0 of the fluid. We will think here about a volume that is fixed relative to a system of coordinates. **Ask class:** if the density is ρ, which may depend on position, what is the mass of fluid in V_0? It's simply $\int \rho\, dV$, over the volume V_0. Now suppose that we define an element of surface area bounding the volume by $d\mathbf{S}$, where $d\mathbf{S}$ is positive along the outward normal to the surface element. Let the fluid velocity at that point be $\mathbf{v}$. **Ask class:** what is the rate (mass per unit time) flowing through this surface element? It's $\rho\mathbf{v} \cdot d\mathbf{S}$, defined so that this is positive if fluid flows out of the volume, and negative if fluid flows into the volume. **Ask class:** what, then, is the total mass flowing out of the volume V_0 in unit time? It is

$$dM/dt = \oint \rho\mathbf{v} \cdot d\mathbf{S} \ , \tag{3}$$

where the integral is over the whole surface. Now, the decrease per unit time in the mass is

$$-\frac{\partial}{\partial t}\int \rho\, dV \ . \tag{4}$$

Setting these two equal we have

$$\frac{\partial}{\partial t}\int \rho\, dV = -\oint \rho\mathbf{v} \cdot d\mathbf{S} \ . \tag{5}$$

Ask class: how can we transform the surface integral into a volume integral by Green's formula? It's the integral of a divergence over the volume, so

$$\oint \rho\mathbf{v} \cdot d\mathbf{S} = \int \nabla \cdot (\rho\mathbf{v}) dV \tag{6}$$

and therefore

$$\int \left[\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho\mathbf{v})\right] dV = 0 \ . \tag{7}$$

This must be true for any volume, so the integrand must be zero and we end up with

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho\mathbf{v}) = 0 \ . \tag{8}$$

This statement of mass conservation is called the *equation of continuity*. What does it mean? It simply says that the change in the mass of a volume in space equals the net amount of mass that flows into or out of the volume! Pretty straightforward. You can write something similar for any quantity that is conserved. For example, suppose you want to know how the total energy in a fixed volume changes. The change must equal the net amount of energy that flows into or out of the volume. Mathematically, this is expressed as the divergence of a quantity (you can think of any nonzero divergence as a source or sink of the appropriate quantity). Note, however, that if a quantity is not conserved then you *cannot* write an equation of this form. For example, consider the number of photons in a volume. This can change spontaneously: think of an atom in an excited state emitting a photon. Therefore, the number of photons in a volume can change even if no photons flow into or out of the volume. If a quantity is conserved, then the flow is called a current, and the general form is

$$\frac{\partial \text{quantity}}{\partial t} + \nabla \cdot (\text{current}) = 0 \; . \tag{9}$$

For fluid flow, $\rho \mathbf{v}$ is called the mass current, or mass flux density.

Conservation of momentum.—This is the $\mathbf{a} = \mathbf{F}/m$ bit. Consider a particular element of fluid. As it moves it is subjected to accelerations because of interactions with the fluid around it. When we discussed hydrostatic equilibrium, we argued that this acceleration would be caused by pressure gradients:

$$\rho d\mathbf{v}/dt = -\nabla p \; . \tag{10}$$

Ask class: how can we check if the negative sign is correct? It says that the acceleration should be opposite to the direction of the gradient of the pressure, i.e., the fluid should accelerate *away* from higher pressures. That makes physical sense.

Ask class: should this equation refer to changes in velocity at a fixed position in space, or changes in the velocity of a moving fluid element? This refers to a moving fluid element, which makes sense when you compare $F = ma$ for, say, a falling object, which clearly follows the object. Therefore, to relate it to the velocity field fixed in space, we need to make a transformation:

$$\partial \mathbf{v}/\partial t + (\mathbf{v} \cdot \nabla)\mathbf{v} = -(\nabla p)/\rho \; . \tag{11}$$

This is called *Euler's equation*. Note that this is just the fluid component of the acceleration, which must be added to any other contributions to the acceleration. For example, if there is a gravitational field $\mathbf{g}$, this must be added to the right hand side. You can confirm that the condition $d\mathbf{v}/dt = 0$ (no acceleration) gives the equation of hydrostatic equilibrium in this case.

In writing the equation like this we have sneakily introduced a simplification. We have assumed that there are no processes of energy dissipation or transfer in the fluid. Therefore, we have assume that there is no viscosity and no conduction. We also assume that there are no particles exchanged between fluid elements. These assumptions constitute the assumption of an *ideal fluid*, and we'll stick to them unless forced to do otherwise :). The reason that, say, viscosity could make a difference is that, colloquially, it introduces "friction" between fluid elements that changes the acceleration. Putting this in gives the Navier-Stokes equation instead of the Euler equation.

Ask class: if there is no heat exchange and no particle exchange, what can be said about the entropy of a fluid element? It's constant; well, it's constant if we assume that each fluid element is in thermodynamic equilibrium at any given moment, which is a standard assumption and is correct if thermodynamic processes are fast compared to fluid motion. Motion with constant entropy is called *adiabatic* motion. Therefore, we have in adiabatic motion that

$$ds/dt = 0 \; . \tag{12}$$

Here we let s represent the entropy per unit mass.

Now let's go on to our final equation.

Conservation of energy.—Assuming no transfer of energy or particles, the energy of a fluid element remains constant. Since the mass is also conserved, the energy per mass also remains constant. **Ask class:** if we ignore internal structure of the molecules that comprise the fluid, what are the contributors to the energy? Kinetic energy, or bulk motion, is one, and internal energy is another. If the fluid is in a gravitational field, then potential energy is another contributor. **Ask class:** what is the kinetic energy per unit mass? It's $\frac{1}{2}v^2$ (non-relativistic, as always). Call the internal energy per unit mass w, the *enthalpy*. Then if the fluid is not in a gravitational field, or is always at the same gravitational potential, we find that along the path of a particle

$$\frac{1}{2}v^2 + w = \text{constant.} \tag{13}$$

If instead the fluid is in a gravitational field,

$$\frac{1}{2}v^2 + w + \Phi = \text{constant} \tag{14}$$

where Φ is the gravitational potential energy per unit mass. For example, in a constant field with acceleration g in the $-z$ direction, $\frac{1}{2}v^2 + w + gz =$constant. These energy conservation equations are called *Bernoulli's equation*.

An alternate way that Bernoulli's equation is sometimes phrased is in terms of *streamlines*. Suppose that the flow is steady, meaning that $d\mathbf{v}/dt = 0$ at any point in the flow. Now define a streamline as a curve that is everywhere in the direction of the velocity vector at that point. You can represent this by

$$dx/v_x = dy/v_y = dz/v_z \; . \tag{15}$$

In steady flow, streamlines correspond to the paths of fluid elements. Therefore, for steady flow, we have $\frac{1}{2}v^2 + w + \Phi$=constant along a streamline (the constant typically takes different values for different streamlines, though). If the flow isn't steady, then streamlines don't necessarily correspond to the path taken by an individual fluid element. Streamlines can't cross each other in steady flow; if they did, there would be strong interaction and time-dependence in the fluid.

Let's conclude by discussing an astrophysical application of fluid dynamics: convection. Convection involves the large-scale motion of fluid and transport of energy by that motion (as opposed to energy transport by motion of individual photons or electrons).

Suppose you have some element of fluid at a higher temperature than the surroundings. **Ask class:** for pressure balance with its surroundings, what does this mean about the density? Density is less, so buoyancy effects cause it to rise. Now it's up further, so the density and pressure of the surrounding medium is less. **Ask class:** if the pressure is less, what happens to the fluid element? It expands until the pressure is in equilibrium with its surroundings. So, **Ask class:** what happens to its density? The density drops. **Ask class:** what is the condition for the element to keep rising? The new, adjusted density needs to still be less than the density of the surroundings, so that buoyancy effects continue to cause the fluid element to ascend. **Ask class:** what does that mean about the new temperature of the element compared to the temperature of its new surroundings? It means that the temperature of the element has to be greater than the temperature of the surroundings.

Now let's think about what this means for energy transport. Suppose that the fluid element rises much more slowly than the speed of sound, so that pressure balance is maintained, but much more rapidly than the time necessary to have heat leak out of (or into) the fluid element. Then the total heat in the fluid element is conserved, and if we can ignore viscosity (which we usually can for this purpose), it means that the element moves adiabatically. This means that the entropy is conserved, so that the temperature gradient is fixed for a given pressure gradient. Call this gradient $\nabla T_{\rm ad}$. Thus, for a given fluid element with an initially small perturbation, we know how its temperature will change as it rises.

Given this, **Ask class:** what is the condition on the gradient of the temperature of the surrounding medium such that the fluid element will continue to rise once perturbed upwards? Since the surrounding temperature has to continue to be smaller than the temperature of the fluid element, it has to drop with increasing height faster than the temperature of the element drops. Therefore, the temperature gradient ∇T must satisfy $\nabla T > \nabla T_{\rm ad}$. This is the Schwarzschild criterion for convection, brought to you by the same person who came up with the Schwarzschild spacetime for uncharged, nonrotating black holes. Another way to phrase this criterion is that for convection to occur the entropy per unit mass must decrease in the outward direction (against gravity), because if a fluid element has higher entropy than its surroundings it will rise. This occurs until enough heat can diffuse out of the element, at which point energy transfer has occurred. In real stars, when convection occurs it is so efficient that the temperature distribution in the star adjusts itself so that convection *barely* occurs.