**Waves and Instabilities**

With the introduction of dynamics in fluids comes the possibility that there is time variation in the density, pressure, etc., of the fluid. In particular, even if you were to set up a fluid in complete equilibrium (e.g., you could imagine a stationary Sun), in reality there are many things that will perturb the fluid slightly. In some cases, the perturbation will be damped away (think of ripples in molasses). In others, it may produce a wave or mode, and in still others it may lead to an instability. The general analysis of such perturbations is broadly useful throughout physics. In this lecture we'll explore some particular examples and their astrophysical applications. First, though, let's sketch out how you would do a perturbation analysis.

- Set up your fluid in equilibrium. You now have the density $\rho_0(\mathbf{r})$, the pressure $P_0(\mathbf{r})$, and other equilibrium quantities, where the "0" subscript indicates equilibrium. Since these equations are equilibrium solutions, they have no time-dependence.
- Write the relevant equations of fluid dynamics for your problem. These will be such that $\rho_0$, $P_0$, etc., satisfy the equations.
- Now imagine that the density, pressure, and so on are changed very slightly from their equilibrium values. That is, say that $\rho(\mathbf{r}) = \rho_0(\mathbf{r}) + \rho_1(\mathbf{r}, t)$, where $\rho_1(\mathbf{r}, t) \ll \rho_0(\mathbf{r})$, and similarly for the pressure or other quantities. Note that we allow the perturbed quantities to depend on time.
- Put these into your equations of fluid dynamics. The lowest order portions (involving just $\rho_0$, etc.) solve the equations automatically, so you can subtract those out. Now, only keep first-order terms in the equation. For example, you would keep $\rho_0 P_1$, but not $\rho_1 P_1$, because that is second-order in magnitude. This is called *linearization* of the equations.
- Now an important trick: just as in Fourier analysis, where any one-dimensional function can be represented by a sum of sines or cosines, here your function $\rho_1(\mathbf{r}, t)$ can be represented by a sum of the functions $C \exp[i(\omega t - \mathbf{k} \cdot \mathbf{r})]$, where $C = C(\omega, \mathbf{k})$ is a constant, $\omega$ is a frequency, and $\mathbf{k}$ is called a wavenumber and has units of $\mathrm{cm}^{-1}$.
- The key is that to linear order, a given $(\omega, \mathbf{k})$ term evolves *independently* from all other $(\omega, \mathbf{k})$ terms. Therefore, we can consider each one of them separately.
- The gist is that you can put $\rho_1 = c_\rho \exp[i(\omega t - \mathbf{k} \cdot \mathbf{r})]$, $P_1 = c_P \exp[i(\omega t - \mathbf{k} \cdot \mathbf{r})]$, and so on into your equations as a trial solution. The time and spatial derivatives are then easy to compute, and you end up with a relation between $\omega$ and $\mathbf{k}$. This is called a *dispersion relation.*
- From the properties of $\omega$, you can figure out whether a perturbation is damped, or oscillates, or grows, or whatever.

This procedure is general enough that we will look at several examples. We'll start with simple ones, where using the full power of this technique is akin to hammering a tack with a sledgehammer, then move to somewhat more complicated examples. However, *never* lose sight of the underlying physics! Dispersion relations are mightily helpful in doing quantitative analysis, but you always need to back it up with an understanding of *why* the instability or oscillation is taking place!

Okay, first example. Consider a pendulum that consists of a mass $m$ at the end of a massless but rigid rod of length $l$, suspended in a constant gravitational acceleration $g$ (downwards). Let the point $O$ be the equilibrium position, and let $s$ measure the distance from $O$. The component of the gravitational force in the direction of $s$ is $-mg\sin\theta$, where $\theta$ is the angle made by the rod to the vertical direction. Therefore, $\theta = s/l$. The equation of motion is

$$m\ddot{s} + mg\sin(s/l) = 0\ . \tag{1}$$

Cancelling the $m$s and writing in terms of $\theta$, we have

$$\ddot{\theta} + \frac{g}{l}\sin\theta = 0\ . \tag{2}$$

Now we linearize. Assume $\theta \ll 1$, so this is a small displacement from vertical. Then $\sin\theta \approx \theta$, so we have

$$\ddot{\theta} + \frac{g}{l}\theta = 0\ . \tag{3}$$

Now we try a trial solution: $\theta = \theta_0 \exp(i\omega t)$. At this point you may wonder: how can the angle be complex? It can't, of course. However, a nice general principle about linear differential equations with real coefficients is that if a complex function satisfies the equation, then the real and imaginary parts must satisfy the equation separately. You'd only get into trouble if there were something that mixed the real and imaginary parts, such as squaring the function. In any case, we'll choose a complex function for convenience of manipulation, then take the real part if necessary for relation to reality.

When we put this trial solution in, we find

$$\begin{gathered} -\theta_0\omega^2 \exp(i\omega t) + (g/l)\theta_0 \exp(i\omega t) = 0 \\ \omega^2 = g/l\ . \end{gathered} \tag{4}$$

This has two solutions: $\omega = +\sqrt{g/l}$ and $\omega = -\sqrt{g/l}$. Either one gives an oscillating solution, with the same result that we find a pendulum swings back and forth! Duh. This is a pure oscillating solution, with no damping or growing terms.

What happens if we consider the same pendulum when it starts in a vertically *upwards* position? By the same logic, we find that when $\theta \ll 1$ the equation of motion becomes

$$\ddot{\theta} - \frac{g}{l}\theta = 0\ . \tag{5}$$

Once again, we try $\theta = \theta_0 \exp(i\omega t)$. Now we find

$$\begin{aligned} -\theta_0\omega^2 \exp(i\omega t) - (g/l)\theta_0 \exp(i\omega t) = 0 \\ \omega^2 = -g/l \ . \end{aligned} \quad (6)$$

That minus sign makes a big difference! It means that $\omega = \pm i\sqrt{g/l}$, so our solution is

$$\theta = \theta_0 \exp(\pm t\sqrt{g/l}) \ . \quad (7)$$

Unlike the oscillating case, where the two solutions were basically equivalent, the two solutions here are *not* the same at all! The one with the negative in the exponential dies away, on a typical time scale $t_0 \approx \sqrt{l/g}$, whereas the one with the positive sign grows, on the same time scale. Therefore, if $\theta_0 \ll 1$ initially, the dying one becomes insignificant whereas the growing one becomes exponentially larger. This is a mathematical statement of something you know from experience: if, say, you balance a ruler vertically on your finger, it is unstable and falls down. This is a pure growing mode, with no oscillating terms.

Therefore, in these simplified examples, when $\omega$ is purely real, you get just oscillation. **Ask class:** what is an astrophysical example of that? Modes on the Sun are long-lived, so they are essentially undamped. When $\omega$ is purely imaginary, you get either growth or damping, depending on the sign. **Ask class:** can they think of an example? The collapse of a supernova core is an instability of this type. In general you can imagine other possibilities. For example, one could have oscillation with damping. **Ask class:** what is an example? Waves produced by a comet impact. For a biological example, think of oscillations of body temperature. These have to be damped, otherwise we could run into a real problem! One could also have an oscillation that grows; this, misleadingly, is called "overdamping"(!). It simply means that the system overcorrects, so the amplitude goes up. Some of you may have experienced this when learning to drive: if you veer to the left, you swing the wheel too far to the right, then even farther to the left, so that you fishtail.

Now let's use this machinery on a problem we studied before: the Jeans instability, where a nearly uniform gas cloud can become gravitationally unstable. Our basic equations are the continuity equation, Euler's equation, and the Poisson equation:

$$\begin{aligned} (\partial\rho/\partial t) + \nabla\cdot(\rho\mathbf{v}) &= 0 \ , \\ (\partial\mathbf{v}/\partial t) + (\mathbf{v}\cdot\nabla)\mathbf{v} &= -(1/\rho)\nabla p - \nabla\Phi \ , \\ \nabla^2\Phi &= 4\pi G\rho \ . \end{aligned} \quad (8)$$

We also need an equation of state, i.e., a relation of pressure to density, temperature, or other parameters. We'll consider an equation of state in which the pressure is just a function of density, which is called a *barotropic* equation of state, $P = P(\rho)$ (for example, in an ideal gas, $P \propto \rho^{5/3}$).

We therefore have four quantities of interest: the density, pressure, velocity, and potential. Let the equilibrium values be represented by a 0 subscript:

$\rho_0(\mathbf{r}),\ P_0(\mathbf{r}),\ \mathbf{v}_0(\mathbf{r}),\ \Phi_0(\mathbf{r})$. Now consider a small time-dependent perturbation:

$$\begin{aligned} \rho(\mathbf{r},t) &= \rho_0(\mathbf{r}) + \rho_1(\mathbf{r},t) \quad ; \quad P(\mathbf{r},t) &= P_0(\mathbf{r}) + P_1(\mathbf{r},t) \\ \mathbf{v}(\mathbf{r},t) &= \mathbf{v}_0(\mathbf{r}) + \mathbf{v}_1(\mathbf{r},t) \quad ; \quad \Phi(\mathbf{r},t) &= \Phi_0(\mathbf{r}) + \Phi_1(\mathbf{r},t) \end{aligned} \tag{9}$$

Now we can place these into the fluid dynamic equations. The zeroth order time-independent terms sum to zero. Therefore, we subtract them out to get the first-order equations (but don't include second-order terms). We'll skip the details. What we need to do is determine the equilibrium state, then put that into the first-order equations so that we can solve for the perturbed quantities. In this situation, this involves what Binney and Tremaine (in their book "Galactic Dynamics") call the "Jeans swindle". The problem is that we are imagining perturbations of an infinite homogeneous medium. If the density and pressure are constant and the mean velocity is zero, this means (from Euler's equation) that $\nabla\Phi_0 = 0$. However, the Poisson equation says that $\nabla^2\Phi_0 = 4\pi G\rho_0$. Oops! The only way these are consistent with each other is if $\rho_0 = 0$, which would be a rather boring situation. The Jeans swindle is to assume that Poisson's equation relates only the perturbed potential to the perturbed potential, and that the unperturbed potential is zero. One must check the validity of this in a given situation, but it usually gives surprisingly accurate answers.

Therefore, we assume an equilibrium state in which $\rho_0$ is a constant in space, $\mathbf{v}_0 = 0$, and $\Phi_0 = 0$. If we use the internal energy per unit mass (also called the enthalpy), we find that the perturbed enthalpy is $w_1 = c_s^2\rho_1/\rho_0$, where $c_s$ is the speed of sound. Our four linearized equations then become

$$\begin{gathered} \partial\rho_1/\partial t + \rho_0\nabla\cdot\mathbf{v}_1 = 0 \\ \partial\mathbf{v}_1/\partial t = -\nabla w_1 - \nabla\Phi_1 \\ \nabla^2\Phi_1 = 4\pi G\rho_1 \\ w_1 = c_s^2\rho_1/\rho_0 \end{gathered} \tag{10}$$

If we take the partial time derivative of the first equation, combine it with the divergence of the second, and eliminate $\mathbf{v}_1$, $\Phi_1$, and $w_1$ in favor of $\rho_1$, we get

$$\frac{\partial^2\rho_1}{\partial t^2} - c_s^2\nabla^2\rho_1 - 4\pi G\rho_0\rho_1 = 0\ . \tag{11}$$

Now we substitute in our trial solution

$$\rho_1(\mathbf{r},t) = C\exp[i(\omega t - \mathbf{k}\cdot\mathbf{r})]\ . \tag{12}$$

Defining $k \equiv |\mathbf{k}|$, this gives us a dispersion relation between $\omega$ and $k$:

$$\omega^2 = c_s^2k^2 - 4\pi G\rho_0\ . \tag{13}$$

Whew! Time to take stock. **Ask class:** what does this mean in terms of stability or instability as a function of the wavenumber $k = 2\pi/\lambda$, where $\lambda$ is the wavelength?

Remember that if $\omega$ is real, we have an oscillating solution, whereas if $\omega$ is imaginary we can have exponential growth. With this in mind, when $c_s^2 k^2 > 4\pi G\rho_0$, we have only oscillations. Note, however, that unlike the case of the pendulum (where the system oscillates in place), here constant phase keeps $\mathbf{k}\cdot\mathbf{r} - \omega t$ constant. **Ask class:** what does that mean, physically? It means that the wave travels. In fact, this is a sound wave! It says that if $k$ is large enough (equivalently, if $\lambda$ is small enough), then squeezing a region of the gas and letting it go produces a wave that travels in the medium at the sound speed. However, if $c_s^2 k^2 < 4\pi G\rho_0$, then $\omega^2 < 0$ and one of the solutions grows exponentially. Therefore, for small enough $k$ (i.e., large enough $\lambda$), a perturbation will grow. The critical wavenumber or wavelength is then given by

$$\begin{aligned} k^2 < k_J^2 &\equiv 4\pi G\rho_0/c_s^2 \\ \lambda^2 > \lambda_J^2 &\equiv \pi c_s^2/G\rho_0 \; . \end{aligned} \tag{14}$$

The mass contained in a sphere of diameter $\lambda_J$ is the Jeans mass:

$$M_J = \frac{4\pi}{3}\rho_0(\frac{1}{2}\lambda_J)^3 = \frac{1}{6}\pi\rho_0\left(\frac{\pi c_s^2}{G\rho_0}\right)^{3/2} \; . \tag{15}$$

Our previous answer, based on energy arguments, was

$$M_J = \left(\frac{5}{2}\frac{kT}{Gm}\right)^{3/2}\left(\frac{4\pi}{3}\rho\right)^{-1/2} \; . \tag{16}$$

This has a numerical factor of 1.9. Since $c_s^2 = 3kT/m$, our new derivation based on the dispersion relation has a numerical factor of 2.7. Pretty close, especially when you consider that perturbations need not be exactly spherical, so there is some slack in the derivation anyway.

I hope this has given you some sense of the mathematics that can go into analysis of the stability of systems. The procedure just followed can be used in a wide variety of circumstances. To be redundant, the key is *always* that you must have a physical understanding of why a system is stable or unstable in some situation. That allows you to catch errors, and also gives you more insight into the system.