Newtonian Gravity

As Liddle points out, a surprising amount of insight can be obtained about the dynamics of the universe using Newtonian gravity. It requires some mild cheating, but since we are aiming for insight we will follow Liddle's treatment in his Chapter 3 and postpone an introduction to general relativity.

We therefore start with the Newtonian gravitational force law

$$F = -\frac{GMm}{r^2} \tag{1}$$

for objects of masses M and m separated by a distance r. Newton's second law $F = ma$ then means that the acceleration felt is the same for any object, independent of its mass. This is a deep statement: the inertial mass (the m in $F = ma$) is equal to the gravitational mass (the m in $F = -GMm/r^2$). Einstein elevated this to the *equivalence principle*, which plays a major role in general relativity. We won't pursue it in this lecture, though.

There is, therefore, a gravitational potential energy for this pair of particles:

$$V = -\frac{GMm}{r} \; . \tag{2}$$

Since there are no negative masses, the gravitational potential energy is always negative, in distinction to, for example, electrostatic potential energy.

We now use a result proved geometrically by Newton. If you have a spherical shell of mass, a particle outside the shell feels a force equal to that from the total mass concentrated at its center, but a particle inside the shell feels no net force.

With this in mind, consider a particle of mass m that is a distance r away from some arbitrary "center" in a homogeneous universe. If the (constant) density is ρ, then the mass contained within that radius r is $4\pi r^3\rho/3$ and the force is

$$F = -\frac{GMm}{r^2} = -\frac{4\pi G\rho r m}{3} \; . \tag{3}$$

The gravitational potential energy is then just

$$V = -\frac{GMm}{r} = -\frac{4\pi G\rho r^2 m}{3} \; . \tag{4}$$

The kinetic energy is $T = \frac{1}{2}m\dot{r}^2$, so the total energy is

$$U = T + V = \frac{1}{2}m\dot{r}^2 - \frac{4\pi}{3}G\rho r^2 m \; . \tag{5}$$

The energy is conserved, so for a given U this tells us how r changes with time.

Again following Liddle, we can now make an important step using the cosmological principle that the universe is homogeneous. This implies that any two points are the same, so we can change to a new set of coordinates called *comoving coordinates*. The idea is that if at any given time a snapshot of the universe looks like a bunch of matter spread uniformly over a grid, then the expansion of the universe means that the grid itself expands, and does so in the same way everywhere (see Figure 1). A particle that passively follows the expansion moves with this expanding grid. Therefore, coordinates that follow the grid are natural ones to choose, and these are the comoving coordinates

$$\vec{r} = a(t)\vec{x} \tag{6}$$

where $\vec{r}$ were the old coordinates (which didn't expand with the universe, and which represent real distances), $a(t)$ is a *scale factor* indicating how big the universe is at any given time, and $\vec{x}$ are our new comoving coordinates.

This is a central concept in cosmology, so let's examine what it means a bit more. Suppose that the Milky Way is currently 1 billion parsecs from galaxy X, and 500 million parsecs from galaxy Y. Let us arbitrarily say that the scale factor is currently $a(t_{\rm now}) = 1$. When the scale factor was $a(t_{\rm then}) = 1/2$, then assuming all the galaxies followed the general expansion of the universe the Milky Way was 500 million parsecs from galaxy X and 250 million parsecs from galaxy Y. **Ask class:** at that time, what was the *comoving* distance of the Milky Way from each of the galaxies? It was still 1 billion and 500 million parsecs, respectively. You'll see similar terms around as well, such as "comoving volume".

Recalling that by definition $\dot{x} = 0$ because those are comoving, this coordinate change gives us

$$U = \frac{1}{2}m\dot{a}^2x^2 - \frac{4\pi}{3}G\rho a^2x^2m\ . \tag{7}$$

If we rearrange and define $kc^2 \equiv -2U/mx^2$ we get

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{kc^2}{a^2}\ . \tag{8}$$

This is the *Friedmann equation*. Note that, of course, the factors of m all cancelled out. You can see that k must be independent of x because everything else is, so that means that $U \propto x^2$. That's okay: we only required U to be independent of time, not of separation. Indeed, since U and x are both independent of time, k must be as well, making it a universal constant in time and space. We'll see later that this is intimately related to the geometry of spacetime.

The Meaning of Expansion

What, though, does this expansion mean? Every now and then I get an e-mail from some crackpot who thinks that he (it's always a man) has some major discovery based on the idea that, for example, since the solar system is about 5 billion years old, it must have been 2/3 its present size at birth (because the universe is about 14 billion years old). Is it the case that everything expands with the universe?

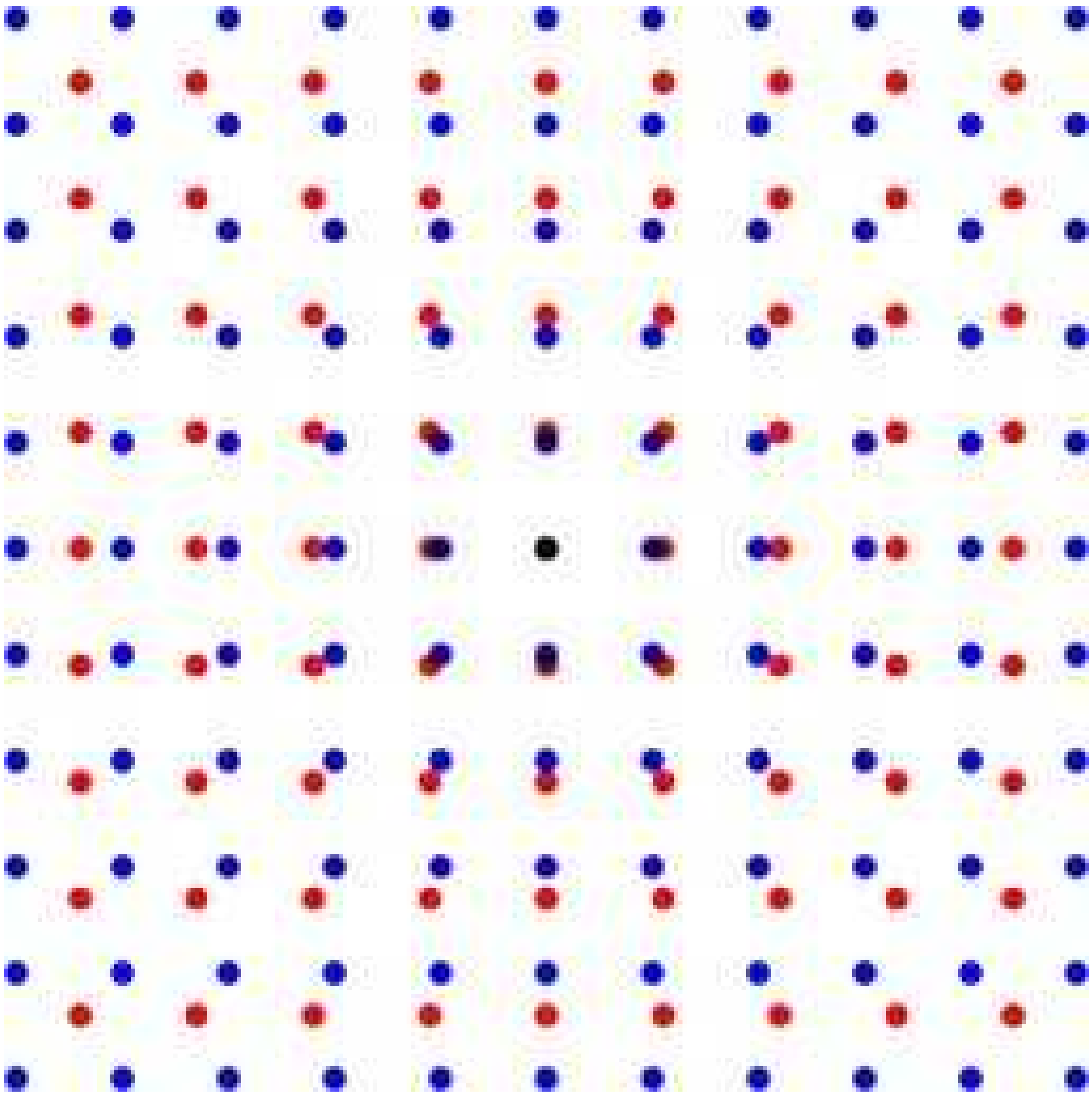

Fig. 1.— Universal expansion, which would look like this from any point in the universe: more distant things appear to move away more rapidly. From http://brahms.phy.vanderbilt.edu/∼rknop/astromovies/dots.jpg

No! Think of it like a moving walkway. Things are carried along the walkway, but local motions are not prohibited and over small enough distances can dominate. Let's do some quantitative calculations.

We will start with the Solar System, and in particular the orbit of the Earth around the Sun. The figure of merit for the expansion of the universe must somehow involve the Hubble constant $H_0 = 72\ \mathrm{km\ s^{-1}\ Mpc^{-1}}$. **Ask class:** what comparison should we make? Here we can be guided by a simple principle, which is that we should try a simple thing first. If we find that either expansion or gravity dominates overwhelmingly, then we're done. Only if they are comparable must we be really careful. With this in mind, one obvious comparison is the orbital speed of the Earth compared to the recession speed that would be implied for universal expansion. If $r = 1$ AU is the orbital radius, then expansion implies a speed of $H_0 r = 3.6 \times 10^{-10}$ km s^{-1}. The Earth's orbital speed is 2πAU/yr, or about 30 km s^{-1}. Universal expansion is clearly unimportant for the Earth's orbit.

How about our galaxy? Say that we are at about 10 kpc from the Galactic center (note the capitalization; it is now common usage to refer to the Milky Way as "the Galaxy"). The Hubble flow would imprint a speed of about 0.7 km s^{-1}, versus our rotational speed of about 200 km s^{-1}. Again not important, but not by as overwhelming a factor.

How about galaxy clusters? The radius of a cluster is about 1 Mpc, implying maybe 70 km s^{-1} from the Hubble flow. The orbital speeds in the clusters are around 1000 km s^{-1}, so again the local motions dominate, but only by an order of magnitude. Going beyond 10 Mpc we find that the two are comparable, and at 1 Gpc the universal expansion clearly prevails. This shows that at local levels, other things matter.

Faster than Light!

There does appear to be a problem with this picture. If the recession rate is proportional to the distance, it means that if you go far enough then the recession rate is faster than the speed of light. Does this really happen?

Yes! This is a great surprise to most people, because it appears to be in conflict with special relativity. It is not, however, so let's try to explain that.

Properly stated, special relativity actually only forbids faster than light motion *measured locally*. It also says that we can never *measure* motion faster than the speed of light, so one consequence of this faster than light motion is that we can never see any signals from it. In fact, this is a resolution of the apparent paradox that (1) the cosmos itself is infinite, yet (2) the part we see is finite. The part we don't see is moving away too fast. You can also work out that if the expansion of the universe is slowing down (i.e., the Hubble "constant" is becoming smaller) then the part of the universe we can see is growing all the time. That is, new material is coming into our horizon as time goes on. If the expansion is speeding up, on the other hand, then material that used to be in our horizon is leaving. As we'll see, this is an important aspect of the inflationary model of the very early universe.

Above all, remember this: every point moving with the Hubble flow is effectively stationary! Redshifts are quoted as speeds because Doppler shifts are more familiar than cosmological redshifts, but those galaxies aren't *really* moving at 200,000 km s^{-1}.

Evolution of Components of the Universe

Given that the universe expands, what happens to the stuff inside it? Before hitting the equations, let's reason things out a bit. Suppose we have some type of fundamental particle and that there are a fixed number of such particles. In that case **Ask class:** how would the number density of the particles evolve? Just as $n \propto a^{-3}$. If you are interested in the mass-energy density of the particles, you then have to determine how the mass-energy per particle changes. A convenient way to do this involves the first law of thermodynamics

$$dE + pdV = TdS \tag{9}$$

where here V is the volume (not the potential energy!), p is the pressure, T is the temperature, S is the entropy, and we assume no particles coming into or going out of our volume. If the mass-energy is

$$E = \frac{4\pi}{3} a^3 \rho c^2 \tag{10}$$

then we have

$$\begin{aligned} dE/dt &= 4\pi a^2 \rho c^2 (da/dt) + (4\pi/3) a^3 (d\rho/dt) c^2 \\ dV/dt &= 4\pi a^2 (da/dt)\ . \end{aligned} \tag{11}$$

Combining these and assuming adiabatic expansion $dS = 0$ gives

$$\dot{\rho} + 3\frac{\dot{a}}{a}\left(\rho + p/c^2\right) = 0\ . \tag{12}$$

What does this mean? First, consider heavy particles moving slowly, such as atoms. Their mass-energy is dominated by their rest mass, which doesn't change as the universe expands, so one would expect $\rho \propto a^{-3}$. Indeed, if we set $p \approx 0$, the equation reads $d\ln\rho/dt = -3d\ln a/dt$, meaning $\rho \propto a^{-3}$. Setting the pressure equal to zero may seem strange, but note that, e.g., for an ideal gas we have $p = nkT = \rho(kT/m)$, so $\rho + p/c^2 = \rho\left[1 + kT/(mc^2)\right]$, so a slowly moving species with $kT \ll mc^2$ has a pressure that can be ignored. However, this is not the case for relativistic species such as photons, for which $p \approx \rho c^2/3$. More on this when we get to Chapter 5 in Liddle.

As shown in the book, one can combine the Friedmann equation with the fluid equation above to get the acceleration equation

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}\left(\rho + \frac{3p}{c^2}\right)\ . \tag{13}$$

This equation shows that for nonrelativistic matter $p \approx 0$ the expansion of the universe slows down. It may be a bit of a surprise that if there is pressure then the slowdown is *greater*. The reason this is surprising is that we are conditioned to think of pressure that forces stuff outward. Note, though, that it is actually pressure *gradients* that do this. At sea level, we experience a pressure of about ten newtons per square centimeter. Over our whole bodies, that's the equivalent of several tons of force, yet we aren't crushed. The reason, of course, is that we have equally large internal pressure opposing this. In the case of the universe, homogeneity means that there are no pressure gradients. Instead, the contribution of pressure (a form of energy density) is to the gravity, which slows things down further. On the other hand, if $p < -\rho c^2/3$ one can get acceleration of the expansion, as appears to be happening now.

A final comment is that from this point on we will start doing things like dropping factors of c, the speed of light. This is a tradition of cosmology, general relativity, and particle physics: work in "natural units" where $c = 1$ and often $G = 1$ and $\hbar = 1$ as well. This means that we can't use units as a way of checking our equations, but it does save some writing. That's the convention, anyway, so we should get used to it.

Intuition Builder

Hold a ball above the ground. What is the magnitude of its gravitational potential energy? Make sure to include the potential energy relative to *all* other bits of mass in the universe. In a homogeneous universe, which is more important: close or distant bits of matter? You should run into a nonsensical answer; why does this problem not rear its head in cosmological calculations?