In this lecture we will continue our discussion of general relativity. We first introduce a convention that allows us to drop the many factors of G and c that appear in formulae, then talk in more detail about tensor manipulations. We then get into the business of deriving important quantities in the Schwarzschild spacetime.

Geometrized units

You may have noticed that Newton's constant G and the speed of light c appear a lot! Partially because of laziness, and partially because it provides insight about fundamental quantities, the convention is to use units in which $G = c = 1$. The disadvantage of this is that it makes unit checking tougher. However, with a few conversions in the bag, it's not too bad.

In geometrized units, the mass M is used as the fundamental quantity. To convert to a length, we use GM/c^2. To get a time, we use GM/c^3. Note, however, that there is no combination of G and c that will allow you to convert a mass to a mass squared (or to something dimensionless), a radius to a radius squared, or whatever. With these units, we can for example write the Schwarzschild metric in its more common form:

$$ds^2 = -(1 - 2M/r)dt^2 + dr^2/(1 - 2M/r) + r^2(d\theta^2 + \sin^2\theta d\phi^2) \; . \tag{1}$$

Tensor manipulations

The metric tensor is what allows you to raise and lower indices. That is, for example, $v_\alpha = g_{\alpha\beta}v^\beta$, where again we use the summation convention. Similarly, $v^\alpha = g^{\alpha\beta}v_\beta$, where $g^{\alpha\beta}$ is the matrix inverse of $g_{\alpha\beta}$: $g^{\alpha\beta}g_{\beta\gamma} = \delta^\alpha_\gamma$, where δ is the Kronecker delta (1 if $\alpha = \gamma$, 0 otherwise). For a diagonal metric (such as Minkowski or Schwarzschild), it is particularly simple: $g^{tt} = 1/g_{tt}$, $g^{xx} = 1/g_{xx}$, and so on (or $g^{rr} = 1/g_{rr}$, etc.), and all off-diagonal components are zero.

Specific energy and specific angular momentum

All that is great, but how does it help us? The most straightforward application emerges from an examination of the components of the four-velocity, u^α or u_α. For this purpose, we will define the concept of a *test particle*. This is useful in thinking about the effect of spacetime on the motion of objects. A test particle is something that reacts to fields or spacetime or whatever, but does not affect them in turn. In practice this is an excellent approximation in GR whenever the objects of interest have much less mass than the mass of the system.

The four-velocity of a particle with mass can be written as $u^\alpha = dx^\alpha/d\tau$, where τ is the proper time (specifically, $d\tau^2 = -ds^2$ in geometrized units). As we showed in an earlier lecture, for *any* spacetime at all, and even in the presence of arbitrary forces, for a particle with mass, the squared four-velocity is $u^2 = -1$ (or $-c^2$ in normal units), and for a photon or other massless particle $u^2 = 0$. By the way, one of the components of the four-velocity may look odd: $u^t = dt/d\tau$. What the heck does that mean? It means the rate of change of *coordinate* time per unit change in *proper* time, which you recall is the time as measured by an observer riding along with the particle.

But what about the four-velocity with lowered indices? In Cartesian coordinates this may not seem to be meaningful, because the metric is diagonal and has +1 for the space components. For example,

$$u_x = g_{\alpha x} u^\alpha = g_{tx} u^t + g_{xx} u^x + g_{yx} u^y + g_{zx} u^z = 0 + u^x + 0 + 0 = u^x \ . \tag{2}$$

The same is true for the y and z components. It is true that $u_t = -u_t$, but that's no big deal.

In spherical polar coordinates, though, this is not the case. Consider the Minkowski line element in such coordinates:

$$ds^2 = -dt^2 + dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \ . \tag{3}$$

The metric is still diagonal, but the components are *not* unity. We can understand this by noting that although in our geometrized system time and distance have the same units, angles are dimensionless and thus the metric coefficients can't be unity. Let's see how this plays out if we consider orbits in the equatorial plane $\theta = \pi/2$ (note that since the system is spherically symmetric, we can always do this). Then, for example,

$$u_\phi = g_{\alpha\phi} u^\alpha = g_{t\phi} u^t + g_{r\phi} u^r + g_{\theta\phi} u^\theta + g_{\phi\phi} u^\phi = 0 + 0 + 0 + r^2 u^\phi \ . \tag{4}$$

Does this remind us of anything in particular? Remember that $u^\phi = d\phi/d\tau$ is the angular velocity. Then $r^2 u^\phi = r(r u^\phi) = rv$, where v is the linear speed in the azimuthal direction. Thus, u_ϕ is the specific angular momentum, where in this context and others "specific" means "per unit mass". Similarly, as it turns out, $e = -u_t$ is the specific energy, where the minus sign comes from our choice of metric signature. Both of these identifications also hold true in Schwarzschild coordinates.

For example, consider a particle in circular motion, although not necessarily Keplerian. Then there is no r or θ motion and $u^2 = -1$ gives us $u^t u_t + u^\phi u_\phi = -1$. We can put this into a more convenient form by writing $(g^{t\alpha} u_\alpha) u_t + (g^{\phi\alpha} u_\alpha) u_\phi = -1$. The Schwarzschild spacetime is diagonal, so this becomes simply $g^{tt}(u_t)^2 + g^{\phi\phi}(u_\phi)^2 = -1$. Consulting the line element, we find $g^{tt} = 1/(1 - 2M/r)$ and $g^{\phi\phi} = 1/r^2$, so the specific energy is

$$e = \sqrt{(1 - 2M/r)(1 + u_\phi^2/r^2)} \ . \tag{5}$$

For example, for a slowly rotating star with $u_\phi \approx 0$, the energy is $e = \sqrt{1-2M/r}$. This means that a particle of mass m originally at infinity will release a total energy $m(1-e)$ if it finally comes to rest on the star's surface. Let's check if this makes sense in the Newtonian limit $r \gg M$. Then $e \approx 1 - M/r$, so the energy released is mM/r, which is the Newtonian form when we put back in the units: GmM/r.

As another example, let's go back to the Minkowski metric and consider a particle that is moving purely in the radial direction. What is its specific energy? Suppose that the radial speed is u^r. Then since $u^\theta = u^\phi = 0$, $u^2 = -1$ gives us

$$\begin{aligned} u^t u_t + u^r u_r &= -1 \\ g^{tt} u_t^2 + g_{rr}(u^r)^2 &= -1 \\ -u_t^2 + (u^r)^2 &= -1 \\ e = -u_t &= \left[1 + (u^r)^2\right]^{1/2} \ . \end{aligned} \tag{6}$$

Does this make sense? For $u^r \ll 1$ (i.e., a speed much less than the speed of light), an expansion gives

$$e \approx 1 + \frac{1}{2}(u^r)^2 \ . \tag{7}$$

Recalling that e is in units of mc^2, this is correct, as it gives the rest mass (the 1 part) plus the kinetic energy. What about very close to the speed of light?

Here's where we have to be careful. In the "lab" frame, being close to the speed of light means that $v = dr/dt$ is nearly 1. However, $u^r = dr/d\tau$, not dr/dt. We therefore have to go the long way around:

$$v = \frac{dr}{dt} = \frac{dr/d\tau}{dt/d\tau} = \frac{u^r}{u^t} = \frac{u^r}{g^{tt}u_t} = \frac{u^r}{-u_t} \ . \tag{8}$$

Therefore, $u^r = -vu_t$. We therefore can rewrite the equation for the specific energy as

$$\begin{aligned} -u_t &= (1 + v^2 u_t^2)^{1/2} \\ u_t^2 &= 1 + v^2 u_t^2 \\ (1-v^2)u_t^2 &= 1 \\ e = -u_t &= (1-v^2)^{-1/2} = \gamma \ . \end{aligned} \tag{9}$$

This is also correct. Note how we had to be precise with our manipulations!

Orthonormal Tetrads

Let's now return to a subject we've mentioned a few times: shifting to a locally Minkowski frame. In general, you want to take a metric that looks like $g_{\alpha\beta}$ and shift into a frame such that locally the metric is $\eta_{\alpha\beta} = (-1, 1, 1, 1)$. It is conventional to represent the new coordinates with hats (e.g., $\hat{t}$, $\hat{r}$, $\hat{\theta}$, $\hat{\phi}$), so that

$$ds^2 = -d\hat{t}^2 + d\hat{r}^2 + d\hat{\theta}^2 + d\hat{\phi}^2 \ . \tag{10}$$

The transformation from the local to the global coordinates is done with the transformation matrices $e^{\hat{\alpha}}{}_{\beta}$ and $e^{\alpha}{}_{\hat{\beta}}$. For example, $u^{\alpha} = e^{\alpha}{}_{\hat{\beta}} u^{\hat{\beta}}$. The components of the transformation matrices come from the transformation of the metric tensor:

$$\eta_{\hat{\alpha}\hat{\beta}} = e^{\mu}{}_{\hat{\alpha}} e^{\nu}{}_{\hat{\beta}} g_{\mu\nu} \ . \tag{11}$$

This is especially easy for the Schwarzschild metric, because the metric is diagonal. Then, for example, $e^{t}{}_{\hat{t}} = (1-2M/r)^{-1/2}$ and $e^{\phi}{}_{\hat{\phi}} = r^{-1}$. Note that even after having transformed into a reference frame in which the spacetime is as Minkowski as possible (i.e., first but not second derivatives vanish), there is still freedom to choose the coordinates. Also, remember that there is always freedom to have Lorentz boosts; that is, having found a frame in which the spacetime looks flat, another frame moving at a constant velocity relative to the first also sees flat spacetime. This means that your choice of frame ("orthonormal tetrad") is based to some extent on convenience. Around a spherical star, a good frame is often the static frame, unmoving with respect to infinity. For a visualization of some of the effects on space and time near a gravitating object, see Figure 1.

Now let's see some examples of this in action. Suppose a particle moves along a circular arc with a linear velocity in the ϕ direction $v^{\hat{\phi}}$ as seen by a static observer at Schwarzschild radius r. What is the angular velocity as seen at infinity? $v^{\hat{\phi}} = d\hat{\phi}/d\hat{t} = u^{\hat{\phi}}/u^{\hat{t}}$. But this is $e^{\hat{\phi}}{}_{\phi} u^{\phi} / \left[e^{\hat{t}}{}_{t} u^{t} \right]$. Since $\Omega = d\phi/dt = u^{\phi}/u^{t}$, then

$$\Omega = (e^{\hat{t}}{}_{t} / e^{\hat{\phi}}{}_{\phi}) v^{\hat{\phi}} = \left(\frac{v^{\hat{\phi}}}{r} \right) (1-2M/r)^{1/2} \ . \tag{12}$$

This makes sense; it's just the same as one would calculate in the Newtonian limit, except that the frequency is less because of redshifting.

With this under our belts, let's do a problem that points out some of the strange things about black holes. Consider a particle of nonzero rest mass that is released from rest a long way from an uncharged, nonspinning black hole (which thus can be described using the Schwarzschild spacetime). At a distance r from the origin, as measured using Schwarzschild coordinates, we want to know (a) what is the proper radial speed, (b) what is the radial speed as measured at infinity, and (c) what is the radial speed as measured by a local static observer?

To start off, we note that because the particle starts at rest at a large distance from the hole, the total energy of the particle is just mc^2, hence $-u_t = 1$. This is a conserved quantity. We also note that because the particle is just falling radially, it means that $u^{\theta} = u^{\phi} = 0$, and $u_{\theta} = u_{\phi} = 0$ as well. We can then work from conservation of the squared four-velocity

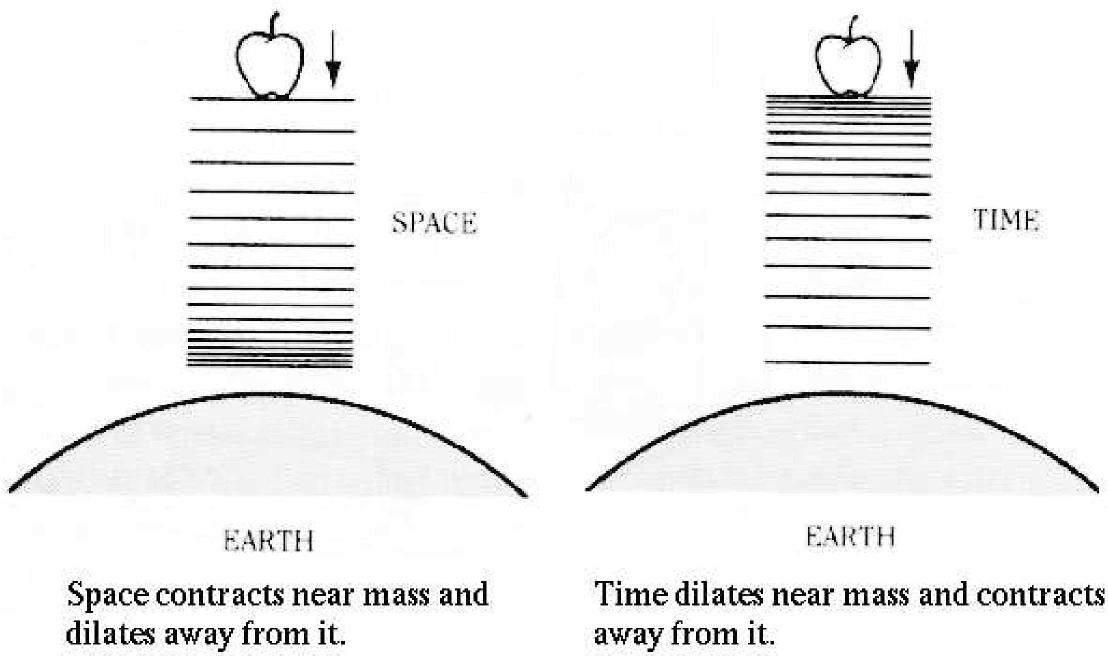

Fig. 1.— Effects on distance and time near a gravitating object. From http://abyss.uoregon.edu/~js/images/spacetime_dilates.gif

to find

$$\begin{aligned}
u^2 &= -1 \\
u^t u_t + u^r u_r &= -1 \\
g^{tt} u_t^2 + g_{rr}(u^r)^2 &= -1 \\
-(1-2M/r)^{-1}(1) + (1-2M/r)^{-1}(u^r)^2 &= -1 \\
(1-2M/r)^{-1}(u^r)^2 &= -1 + (1-2M/r)^{-1} \\
(u^r)^2 &= -(1-2M/r) + 1 \\
u^r &= -(2M/r)^{1/2} \; .
\end{aligned} \tag{13}$$

The negative sign is because the particle is falling inward. If we were to put the units back in, this would be $-(2GM/r)^{1/2}$.

To interpret this, we need to recall the meaning of proper speed, etc.

(a) Remember that "proper" means "as measured by a comoving observer". Therefore, the proper radial speed is $dr/d\tau$. This is just u^r, so the answer to the first part of our question is that the proper radial speed is $u^r = -(2M/r)^{1/2}$.

(b) What is the radial speed as measured at infinity? Recall from the definition of Schwarzschild coordinates that the coordinate time t is the time as measured at infinity. Therefore, the radial speed as measured at infinity is dr/dt. To relate this to the proper radial speed we need to do the following:

$$\begin{aligned}
dr/dt &= (dr/d\tau)/(dt/d\tau) \\
&= u^r/u^t \\
&= u^r/(g^{tt}u_t) \\
&= u^r/[-(1-2M/r)^{-1}(-1)] \\
&= (1-2M/r)u^r \\
&= -(1-2M/r)(2M/r)^{1/2} \; .
\end{aligned} \tag{14}$$

(c) What is the radial speed as measured by a local static observer at r? For this, we need to transform between the global four-velocity components (e.g., u^r, u^t) to the local four-velocity components in an orthonormal tetrad (e.g., $u^{\hat{r}}$, $u^{\hat{t}}$). Remember the rule: the transformation matrices take your metric and turn it into a Minkowski metric. For example, consider the tt part of the Schwarzschild line element: $g_{tt} = -(1-2M/r)$. To turn this into $g_{\hat{t}\hat{t}} = -1$, we need to multiply this by $e^t{}_{\hat{t}} e^t{}_{\hat{t}}$, which tells us that $e^t{}_{\hat{t}} = (1-2M/r)^{-1/2}$. Similarly, to go back, it must be that $e^{\hat{t}}{}_t = (1-2M/r)^{1/2}$. Armed with this information, we then find

$$\begin{aligned}
d\hat{r}/d\hat{t} &= u^{\hat{r}}/u^{\hat{t}} \\
&= e^{\hat{r}}{}_r u^r/[e^{\hat{t}}{}_t u^t] \\
&= [e^{\hat{r}}{}_r/e^{\hat{t}}{}_t] u^r/u^t \\
&= [(1-2M/r)^{-1/2}/(1-2M/r)^{1/2}] dr/dt \\
&= (1-2M/r)^{-1}[-(1-2M/r)(2M/r)^{1/2}] \\
&= -(2M/r)^{1/2} \; .
\end{aligned} \tag{15}$$

Intuition Builder

In the last problem we did, there are several apparent anomalies. Indeed, these anomalies are the source of much confusion and much spewing by crackpots. They are: (a) the proper radial speed seems to become greater than 1, i.e., greater than the speed of light, when $r < 2M$, (b) the radial speed measured at infinity seems to go to zero as $r \to 2M$, and (c) the radial speed measured by a local static observer appears to become greater than the speed of light when $r < 2M$. What is going on in each case?