ASTR 601 Problem Set 2: Due Thursday, September 28

1. In this problem we will gain insight about the Einstein relations and detailed balance by considering a three-level atom. That is, we will assume that the atom has levels 1, 2, and 3, with respectively n_1, n_2, and n_3 atoms in each level, and that the statistical weights are respectively g_1, g_2, and g_3. Here 1 is the ground state, 2 is the first excited state, and 3 is the second excited state, so that the energy of level 3 is greater than that of level 2, which is greater than that of level 1. Detailed balance means that every microscopic process is in balance with its inverse. That is, in equilibrium and assuming detailed balance, the rate (number per time) at which atoms in level 1 go to level 3 is the same as the rate at which atoms in level 3 go to level 1, and similarly for the other two combinations. With that assumption, each pair of levels acts like a two-level atom and we recover the Einstein relations even if we have more than two levels in the atom.

But here we will relax that assumption. Instead, we will impose the weaker condition that the rate at which atoms move out of level 1 (into *either* level 2 or level 3) is the same as the rate at which atoms move into level 1 (from *either* level 2 or level 3). This guarantees that the number in each level is constant. With that in mind:

a. Write the three equations that guarantee constancy of numbers of atoms in each level: what is the equation for level 1, what is the equation for level 2, and what is the equation for level 3?

b. Demonstrate that it is possible to satisfy these equations while also violating detailed balance. Because the Einstein relations rely on detailed balance, this means that the Einstein relations would not hold in this case.

2. The center of the Sun is estimated to have a density of $\rho = 150$ g cm^{-3} and a temperature of $T = 1.5 \times 10^7$ K. Assume that the composition of matter at the Sun's center is pure, fully ionized hydrogen. Say that the thermal energy per particle is $E_{\rm th} = (3/2)kT$, and remember that at a number density n, the Fermi momentum is $p_F = (3\pi^2 n)^{1/3}\hbar$. To within 1%, calculate the ratio of the Fermi energy to the thermal energy for the protons, and separately for the electrons. **Hint:** do you need to include relativistic effects when you compute the Fermi energy? Based on your calculations, are the protons and/or the electrons degenerate? Strongly degenerate?

3. Again diving into our practice problems, write a code to compute, and plot, the ionization fraction y over a range of temperatures from $T = 10^3$ K to $T = 10^6$ K, in logarithmic steps $d\log_{10} T(\mathrm{K}) = 1$ for $\rho = 10^{-31}$ g cm^{-3} (roughly the average baryon density of the universe, $\rho = 10^{-24}$ g cm^{-3} (representative of the average density of the interstellar medium), and

$\rho = 10^{-16}$ g cm^{-3} (a reasonable density for a core of a molecular cloud). What trends do you see, and how would you explain them? **Please send me a copy of your code before you hand in the plots on the due date. Any language is fine as long as it compiles and runs on my departmental machine (please send me compilation/run instructions); I won't install any libraries or download modules!** Please ensure that when your code runs, it produces both a plot of $\log_{10} y$ versus $\log_{10} \rho$ for each temperature (which is all you need for the hardcopy) and a table of $\log_{10} y$ versus $\log_{10} \rho$ for each temperature. In the table, your values of $\log_{10} y$ must be correct to at least three significant figures. To do that, please use the high-precision version of the Saha equation

$$\frac{y^2}{1-y} = \frac{4.0355 \times 10^{-9}}{\rho} T^{3/2} e^{-1.57887 \times 10^5 / T} \ , \tag{1}$$

where ρ is the *total* mass density (including protons) in g cm^{-3} and T is the temperature in K. **For this problem set only, if you get drafts of your codes to me well in advance then I will give you hints about whether your numbers are accurate. From this point on, however, I will simply acknowledge receipt of the code and indicate whether it ran, if I get it early enough before class.**

4. Dr. Sane has discovered a fundamental inconsistency in classical electromagnetism that has been covered up by establishment scientists. If you solve Maxwell's equations you find that in the presence of matter, electromagnetic waves travel at speeds slower than c. But light is supposed to travel at a universal speed! Given this, all of special relativity, including classical electromagnetism, falls apart. Dr. Sane has sent out a press release to this effect, and science writer Ron Cowen has contacted you for your opinion. Your opinion has to be more elaborate than "that only works in vacuum": *why* is it reasonable that light travels slower in a medium?