ASTR 601 Problem Set 3: Due Thursday, October 19

1. (4 pts) Dr. Sane has turned his penetrating intellect to the study of electrons and photons in free space. He has submitted two manuscripts to Physical Review Letters, and you have been asked to evaluate his analyses.

(a) In his first paper, he finds that freely moving (non-accelerated) electrons in a vacuum will emit radiation. Using a conceptual argument, and also using four-momenta, evaluate whether $e^- \to e^- + \gamma$ is possible in free space. Recall that for a particle with nonzero rest mass m, the four-momentum is m times the four-velocity, and that for a photon, the four-momentum is $P^\alpha = (E/c, p_x, p_y, p_z)$ in Cartesian coordinates, where $E = pc$ if $p = \sqrt{p_x^2 + p_y^2 + p_z^2}$. **Hint:** you can save yourself considerable effort by *squaring* the four-momenta on either side of the equation and using the invariance of the squares for the electrons and for the photon.

(b) In his second paper, he investigates photon splitting. In this process, a single photon moving in a vacuum can split into two photons. Suppose that, as seen in some frame, the original photon has energy E. In that same frame, the two resulting photons have energies E_1 and E_2 and move at angles θ_1 and θ_2 relative to the original direction (the three photons all move in the same plane). Derive the conditions on E_1, E_2, θ_1, and θ_2 for splitting to occur, using four-momenta. Discuss whether the process is possible. **Hint:** the argument is somewhat more subtle than it is in part (a).

2. (8 pts) This is a numerical problem. Suppose that you have an isotropic source of initially unpolarized light in a medium, and that the source is a distance $h = 1$ below an infinitesimally thin, and therefore two-dimensional, planar scattering layer. If light travels a distance d in the medium, its probability of *not* being absorbed is e^{-d}. We will consider only the *unabsorbed* light in this problem. Suppose that when that light reaches the scattering layer (which we assume to extend indefinitely in both directions), it undergoes one Thomson scattering; if it scatters back into the medium it absorbs and we don't follow it. The scattered light (from anywhere in the disk) is measured by an infinitely distant observer looking at an angle θ with respect to the disk surface (see the figure). Photons travel in straight lines.

Your task is to write a code to compute the net linear polarization seen by the observer as a function of θ, and to determine the direction of that polarization. The easiest way is to use the Stokes parameters. As usual, I need you to send me your code (in any language but in a form that can be compiled [please send instructions!] and run on my departmental desktop) before the class starts. In addition to producing a figure with the net polarization fraction as a function of θ (which is all you need to submit as your hardcopy), your code must print out a table of polarization fraction versus θ to at least three significant figure accuracy.

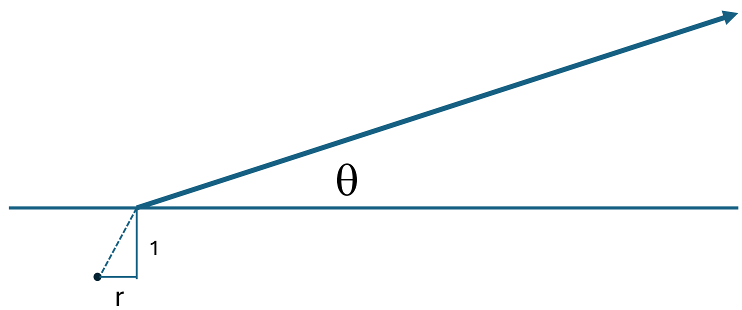

Fig. 1.— Geometry for Problem 2. The small dark ball indicates the starting position of the photon. It begins 1 unit of distance beneath the scattering layer, which is represented by the horizontal line. This photon moves at a slight angle, hits the scattering layer (without first being absorbed) a horizontal distance r from its original location, travels in a straight line at an angle θ with respect to the scattering layer, and eventually reaches the distant observer.

This is a problem with several components and therefore several ways to go wrong. Please answer the following sequence of questions, which I hope will help guide you through your coding:

(a) What is the angle of scattering α for a given r, ϕ (azimuthal angle that goes along with r; you'll discover you need it), and θ? $\alpha = 0$ for a photon that moves in a straight line from the source to the observer. To help set you up, please remember that the easiest way to get the angle between two unit vectors is to compute their dot product, which is the cosine of that angle. We therefore need to start by setting up a coordinate system. Let the $+z$ axis be the one going from the source to the point on the scattering surface directly above it. Let the $+x$ axis lie in the scattering surface, and be in the plane formed by the source, the point directly above the source, and the observer. Let the origin of the coordinate system be at the source. Then, in standard spherical polar coordinates, the observer is at $\phi = 0$ and the angle from the z axis is $\psi = \pi/2 - \theta$.

(b) What are the right formulae for the polarization fraction and Thomson cross section as a function of α? Recall that although the direction-integrated Thomson cross section is σ_T, which is a constant, the *differential* cross section for Thomson scattering depends on the angle between the initial direction and the final direction after scattering.

(c) How much light (energy per time) arrives, unabsorbed, at a differential element of the scattering layer (you can think about this as a small square in the plane of the scattering surface) between angles ϕ and $\phi + d\phi$, and between distances r and $r + dr$ away, from the point straight above the source? Note that if the direction of the light is anything other than straight up, then there is an angle between the initial direction of propagation and the normal to the surface element. In addition, the farther the light goes in the medium, the less that can get through, because of the optical depth.

(d) Remember that the Stokes parameters add linearly, which means that you can loop over all rays and add the correct amounts to V, I, Q, and U.

(e) There is at least one limit, and at least one symmetry, that you can use to check your answer. Does your answer satisfy those checks?

Good luck!

3. (4 pts) Consider a situation in which photons are only allowed to have head-on collisions with electrons; that is, they hit and bounce straight back. The photons are put in a one-dimensional tube in which the speed of the electrons is drawn from a Gaussian, $P(v) \propto e^{-m_e v^2/(2kT)}$, where m_e is the electron mass, T is the temperature, and we assume that the speed is much less than the speed of light: $v \ll c$. The electrons have an equal probability of moving left or right, and we assume that in the electron rest frame the photon energy is much less than $m_e c^2$, which implies that the scattering cross section is always the Thomson cross section $\sigma_T = 6.65 \times 10^{-25}$ cm^2. You'll run into some nasty integrals; please make these easier on yourself by expanding v/c to whatever is the lowest remaining order (warning: this might not be linear order). Your task is to use the principles of Compton scattering to determine the energy $E_{\rm equil}$ of photons such that the average energy of the photons after scattering is still $E_{\rm equil}$. Remember that the rate of interactions, as seen in the rest frame (where the electron velocity distribution is left-right symmetric), depends on the relative speed between the photons and the electrons; if the electron is moving away from the photon, the interaction rate is less than if the electron is moving towards the photon.