Units, limits, and symmetries

When solving physics problems it's easy to get overwhelmed by the complexity of some of the concepts and equations. It's important to have ways to navigate through these complexities and reduce errors. One of the best navigation tools is a sense of what the answer should look like. What units should it have? How should it behave in easily-understood limits? What are the symmetries of the problem? What should the answer depend on? You should check every answer you get against these common-sense guides. This will cut down dramatically on errors in derivation. Even more importantly, it will help build up your intuition about physics, because you will be able to approach problems by constraining the answer first. It will also put you one step ahead of quite a number of researchers; you'd be amazed how often you'll be able to get to an answer quickly using these techniques!

In this class we'll run into a *lot* of equations, some of them pretty hairy. I want you to form the habit of checking each equation for units, limits, and symmetries (as well as other things such as conservation laws). To help with this, Doug Hamilton and I are writing a book of practice problems. In a given problem, you're presented with a physical situation and several possible expressions for the correct answer. The point it *not* to get the right answer per se, but to be able to rule out definitely wrong answers using simple arguments. In our radiative processes class, I'll frequently stop and consider an equation to see if it can be the right one.

For our first lecture, then, I want to give examples of this and practice on a few cases in mechanics (to give the idea), then radiative processes. To do this I am taking examples from the book Doug and I are writing, as well as some of the introductory material.

1. CHECKING UNITS

Units are the first thing to check when considering possible answers to a problem. Any equation that you write must be dimensionally correct. Check your equations occasionally as you go through a derivation. It takes just a second to do so, and you can quickly catch many common errors. Remember this general rule: in all physically valid solutions, the argument of all functions (e.g. trigonometric functions, exponentials, logs, hyperbolic functions, etc.) must be dimensionless. Taking the cosine of something with units of mass or length makes no physical sense. For each of the problems below, imagine that you and several friends have just gone through lengthy derivations and each come up with different answers. In each case, you can rule out several of the answers with dimensional arguments without ever having to look at the actual derivations.

Here's an example:

1. A daredevil is shot out of a cannon at speed v and angle θ from horizontal. Earth's gravitational acceleration, g, is assumed constant, and air resistance is neglected. How far downrange, D, does the daredevil fly before hitting the ground?

A) $D = 2v^2 \cos\theta$
B) $D = (2v^2/g) \sin\theta$
C) $D = 2g \sin\theta \cos\theta$
D) $D = 2vg(\cos\theta - \sin\theta)$
E) $D = (2v^2/g) \sin g$

Answer: Distance is measured in meters, velocities in m/s, and acceleration in m/s^2. All of the above answers have left-hand sides which are distances in units of meters, so the correct answer must have units of meters on the right as well.
A) has units of velocity squared (WRONG)
B) has units of meters (COULD BE OK)
C) has units of acceleration (WRONG)
D) has units of meter squared per second cubed (WRONG)
E) the argument of the sine has units (WRONG)
Note that units checking like this is important but does have limitations. For example, any equation that is dimensionally correct is also dimensionally correct if either side is multiplied by an arbitrary dimensionless factor.

2. CHECKING LIMITS

Check all of your final answers and important intermediate results to see if they behave correctly in as many different limits as you can think of. Sometimes you will know how a general expression should behave if a particular variable is set to zero, infinity, or some other value. Make sure that your general expression actually displays the expected behavior!

Here is a simple example that can be used an a mnemonic for some trigonometric multiple angle formulae.

2. The double angle formula for sines is $\sin(2\theta) = 2\sin\theta\cos\theta$. Which of the following expressions might be correct generalizations?

A) $\sin(\theta_1 + \theta_2) = \sin\theta_1 \cos\theta_2 - \cos\theta_1 \sin\theta_2$
B) $\sin(\theta_1 + \theta_2) = \sin\theta_1 \cos\theta_2 + \cos\theta_1 \sin\theta_2$
C) $\sin(\theta_1 + \theta_2) = \cos\theta_1 \cos\theta_2 + \sin\theta_1 \sin\theta_2$
D) $\sin(\theta_1 + \theta_2) = \cos\theta_1 \cos\theta_2 - \sin\theta_1 \sin\theta_2$

Answer: We know how the correct equation must behave in a number of limits. If $\theta_1 = \theta_2$, for example, the correct expression must reduce to the formula for $\sin(2\theta)$ given above.
A) reduces to $\sin(2\theta) = 0$ (WRONG)
B) reduces to $\sin(2\theta) = 2\sin\theta\cos\theta$ (COULD BE OK)
C) reduces to $\sin(2\theta) = 1$ (WRONG)
D) reduces to $\sin(2\theta) = \cos^2\theta - \sin^2\theta$ (WRONG)
Note that the equation is wrong if it fails for *any* value of θ - so C) and D) are wrong because they fail for $\theta = 0°$ while A) is wrong because it fails for $\theta = 45°$. Note that you can obtain the formula for the sine of the difference of two angles by letting $\theta_2 \to -\theta_2$ in B).

3. TAKING ADVANTAGE OF SYMMETRIES

Symmetries are fundamental in physics (and astronomy!). Problems can have symmetry about a point (spherical symmetry), a line (cylindrical or axial symmetry), or a plane (mirror symmetry). You can use symmetries in two ways: 1) to check your final answer to a problem or, with a little more effort, 2) to simplify the derivation of that final answer. As an example, time-independent central forces (like gravity) have spherical symmetry because the force depends only on the distance from the origin. In this case, spherical symmetry means that once we find one solution (e.g. a particular ellipse for gravity), all other possible orientations of this solution in space are also solutions.

Another type of symmetry could be called a symmetry of labeling. In many problems, it is clear that simply renaming two identical things can't change anything fundamental about the system. For example, consider two objects of mass m_1 and m_2 moving in circular orbits around each other, bound by gravity, separated by a distance a. What is the frequency of rotation? A guess like $\omega = \sqrt{G(2m_1 + m_2)/a^3}$ can't be right, because the answer would change simply by switching the labels on the masses.

4. PUTTING IT TOGETHER

Now let's use these for some more complicated problems. First we'll use a mechanics problem, then several problems related to radiative processes. For these, have class rule out one at a time (that is, ask for someone to rule out one answer; then another student rules out another answer; and so on).

3. You launch a rocket straight up from the Earth's North pole, and it rises up to a maximum height H, then falls back to Earth. The maximum height above the Earth is given by

one of the expressions below. Here R_E is the Earth's radius, $X = v^2 R_E/GM_E$, G is the gravitational constant, M_E is the Earth's mass and v is the launch velocity. Rule out as many of the following expressions that you can.

A) $H = R_E X/(1 + \sqrt{X})$
B) $H = R_E X/(1 - X)$
C) $H = R_E X/(2 - X)$
D) $H = R_E (1 - X)/(2 - X)$
E) $H = vX^2/(2 - X)$
F) $H = R_E X/2$
G) $H = R_E X^2/(2 - X)$
H) $H = R_E X|1 - X|/(2 - X)$

Answer: A key observation is that X is dimensionless. Therefore, any expression involving just X is also dimensionless. We want a height, so E) has the wrong units. In the limit $v \to 0$, $H \to 0$, so D) is incorrect. If v is small but nonzero, then conservation of energy $\frac{1}{2}mv^2 = mgH$ means that $H = v^2/2g = v^2 R_E^2/(2GM_E) = R_E X/2$, since at the surface $g = GM_E/R_E^2$. Small v means $X \ll 1$, so in that limit we need $H = R_E X/2$. This eliminates A), B), and G). We also know that when v is the escape velocity $v_{\text{esc}} = \sqrt{2GM_E/R_E}$, H must become infinite. That means that when $X \to 2$, $H \to \infty$. This eliminates F). We also note that H) predicts the height will go to zero when $X \to 1$, which is incorrect so it, too, is wrong. Our only remaining answer is C).

4. A particle with electric charge e moving in a vacuum at velocity v is accelerated at a rate a. We want to know the expression for the power radiated by the charge.

A) Is P proportional to e or e^2?
B) Is P proportional to v? v^2? a? a^2?

Answer: We know that power is non-negative (the charge can't take energy from the vacuum!). Since e, v, and a can all be negative, this means that P can't depend on an odd power of any of them. Thus, if P depends on e it must be as e^2 given the choices. How about velocity vs. acceleration? Consider a limiting case: constant velocity ($a = 0$). If the power depends on the velocity, then one observer sees a nonzero power, whereas another (comoving with the particle) sees zero power. This kind of disagreement is impossible. Therefore, the power can't depend on v^2. How about a^2? This is a quantity that inertial observers agree on, so it can be the correct answer.

5. The plasma frequency Ω_e is a frequency such that waves in a plasma with less than this

frequency are attenuated rapidly, whereas waves with greater than this frequency can propagate with comparatively little loss. Which of the following could be the plasma frequency? Here m_e is the mass of the electron and N_e is the number density (number per cm^3) of free electrons.

A) $\Omega_e = 4\pi N_e e^2/m$
B) $\Omega_e = \sqrt{4\pi N_e e^2/m}$
C) $\Omega_e = m/(4\pi N_e e^2)$
D) $\Omega_e = \sqrt{m/(4\pi N_e e^2)}$

Answer: Using e^2/r (units of energy) as a basis, we find that only B) has the required units of s^{-1}.

6. A photon of wavelength λ scatters off of a particle of mass m that is initially at rest. After scattering, the photon travels in a direction that is an angle θ from its initial direction. The final wavelength λ_f of the photon is related to the initial wavelength through the relation

$$\lambda_f - \lambda = \pm\lambda_c(1 - f(\theta)) \ , \tag{1}$$

where λ_c is the Compton wavelength.

A) Should the overall sign of the right hand side be $+$ or $-$?
B) Is $f(\theta) = \sin\theta$ or $f(\theta) = \cos\theta$?
C) Is $\lambda_c = h/mc,\ c/hm$, or m/hc?

Answer: A) If the particle was at rest, then the photon can only lose energy so that the final wavelength is greater than the initial wavelength. Therefore, the overall sign must be positive. B) If $\theta = 0$ there was no deflection, so the final wavelength must be equal to the initial wavelength. Thus, $f(\theta) = \cos\theta$. C) The units of h may be obtained from a known formula, such as the energy being $h\nu$ for a frequency ν. Thus, h has units of erg-s and $\lambda_c = h/mc$.

7. In Bohr's semiclassical model of the hydrogen atom, it is assumed that the electron moves in a circle around the nucleus. In the ground state of hydrogen, the angular momentum of the electron equals Planck's constant $\hbar$. The electrostatic potential energy of charges q_1 and q_2 separated by distance r is $q_1 q_2/r$. If an electron of charge e and mass m orbits around a nucleus of charge Ze that is assumed fixed, which of the following could be the binding energy of the atom in its ground state?

A) $E = me^2c/\hbar$.
B) $E = Z^2mc^2$.

C) $E = Z^2 me^4/(2\hbar^2)$.
D) $E = me^4/(2Z^2\hbar^2)$.

Answer: Greater charge means greater electrostatic attraction, so the energy has to increase with increasing Ze. This eliminates A), B) (which depends only on Z but not e), and D). Only C) might be right.

8. Consider two diatomic molecules. The molecules are identical with each other (i.e., their atoms are the same) except that the isotopes of the atoms in the first molecule differ from those in the second. If the masses of the atoms in the first molecule are m_1 and m_2, and in the second are m_1' and m_2', what is the ratio of the oscillation frequencies ω and ω' of the molecules? Note that the "spring constants" of the two molecules are equal to each other.

A) $\omega'/\omega = \sqrt{m_1 m_1'/(m_2 m_2')}$.
B) $\omega'/\omega = \sqrt{m_1 m_2}$.
C) $\omega'/\omega = \sqrt{m_1 m_2 (m_1' - m_2')/[m_1' m_2' (m_1 + m_2)]}$.
D) $\omega'/\omega = \sqrt{m_1 m_2 (m_1' + m_2')/[m_1' m_2' (m_1 + m_2)]}$.

Answer: The ratio must of course be dimensionless, which eliminates B). If $m_1 = m_1'$ and $m_2 = m_2'$, the ratio must be 1, which eliminates C) and A) (note that m_1 could be different from m_2). Only D) has the proper symmetries and limits.

9. A molecule with moment of inertia I can only undergo a rotational transition if the temperature is high enough. What is that critical temperature? Here k is the Stefan-Boltzmann constant, so that thermal energy is kT.

A) $T_{\rm crit} = \hbar^2/(kI)$.
B) $T_{\rm crit} = \hbar k I$.
C) $T_{\rm crit} = I/k$.
D) $T_{\rm crit} = kI/\hbar$.

Answer: From the hint, k has units of J K^{-1} =kg m^2 s^{-2} K^{-1}. No other quantity in the problem has units of K, so to get a final answer in Kelvin we know k must be in the denominator. That eliminates B) and D). Since I has units of kg m^2, C) also gives the wrong units. Only A) is a possible answer.

10. The relativistically-correct expression for adding two velocities is:

A) $v_{total} = (v_1 + v_2)/(1 + v_1 v_2/c^2)$
B) $v_{total} = (v_1 + v_2)/(1 - v_1 v_2/c^2)$

C) $v_{total} = (v_1 + v_2)/(1 + v_1^2/c^2)$
D) $v_{total} = (v_1^2 + v_2^2)/c(1 + v_1 v_2/c^2)$
E) $v_{total} = (v_1 - v_2)/(1 + v_1 v_2/c^2)$

Answer: The expression must be symmetric between the indices 1 and 2, so C) and E) are wrong. If either velocity goes to zero, the total velocity is the other velocity (e.g., if $v_1 = 0$, $v_{total} = v_2$). This eliminates D). What if $v_1 = c$? Then formula B) predicts that the total velocity is $-c$, so that the direction is reversed! This is absurd, so we can eliminate B) as well, leaving A) as the only possibility.

11. Many derivations in quantum mechanics can be checked by letting $\hbar \longrightarrow 0$ and making sure that the answer corresponds to the classical limit. Consider a simple harmonic oscillator, which has a frequency $\omega = 2\pi\nu$ and mass m. Which of the following could be the possible energy states for the oscillator? Here n is a nonzero integer, $0, 1, 2, \ldots$.

A) $E_n = \frac{1}{2}\hbar\omega$.
B) $E_n = (n + 1/2)\hbar\omega$.
C) $E_n = (n + 1/2)m^2c^4/(\hbar\omega)$.
D) $E_n = n\omega$.

Answer: D) has the wrong units. Let $\hbar \to 0$. Then A) predicts the energy is always tiny, and C) predicts that the energy is always huge. Both are wrong. B) is the only possibility.

Finally, let's talk a bit about creativity in science. Obviously we can't reduce this to an algorithm, but there is an approach that can be useful in many circumstances. The idea is that if you are trying to come up with an idea (of how to observe something, or construct an instrument, or of the correct model for some phenomenon) you divide it into two steps: (1) brainstorming, and (2) critical evaluation of the ideas. In step (1), you write down anything that comes to mind; don't self-censor at this point. After you have exhausted that list, take a break, then come back and try to kill as many ideas as you can. Mind you, they have to be clean kills; for example, if one of your ideas is short by a factor of a million in energy, that's clean, but if the apparent disagreement is in the details then you might not want to eliminate it just yet. The idea of this approach is that you have the best chance of coming up with something truly innovative, while also remaining grounded in reality.

As an example for you to try, what powers the Sun? We actually know that it is nuclear fusion, but pretend you don't know that. Come up with as many fundamental power sources as you can (in the brainstorming step). When you can't think of any more, then see how many you can kill from the requirements that (a) the Sun has emitted around 5×10^{50} ergs thus far in its lifetime, and (b) the Sun does so in a stable manner. Interestingly, when I have performed this exercise with many classes, a large number of the alternative mechanisms they considered actually are a major or even dominant source of energy for other astronomical objects.