Cyclotron and Synchrotron Radiation

Initial questions: What is one way that we can get a power law spectrum in astronomy? What information can we obtain? How can we measure the magnetic fields of individual objects? How can we determine that a particular spectrum is produced by synchrotron radiation rather than other effects?

When magnetic fields are present, charges can interact with them and radiate or absorb radiation. For slowly moving particles this happens at a single frequency, the cyclotron frequency. For relativistically moving particles, the emission or absorption occurs over a large range of frequencies, and is called in this case synchrotron radiation. Both names refer to laboratory accelerators.

Let's go back to basics, and assume that we have a charge in a region that has a magnetic field but no electric field. **Ask class:** from the electromagnetic fields alone, does the particle energy change? No, because the acceleration due to a magnetic field is perpendicular to the motion. Therefore, a particle of Lorentz factor γ will evolve in momentum and energy by

$$\begin{aligned} \frac{d}{dt}(\gamma m\mathbf{v}) &= \frac{q}{c}\mathbf{v}\times\mathbf{B} \\ \frac{d}{dt}(\gamma mc^2) &= q\mathbf{v}\cdot\mathbf{E} = 0\ . \end{aligned} \tag{1}$$

Ask class: what does this mean about the total speed? Constant, of course. **Ask class:** how does the speed along the magnetic field change? It doesn't, because the acceleration is all perpendicular to magnetic field. Since the total speed is constant and the speed along the field is constant, this implies circular motion around the field, combined with (possibly) a uniform drift along the field. That is, the charge moves in a helix along the field. If we write just the perpendicular component of the equation of motion, we get

$$\frac{d\mathbf{v}_\perp}{dt} = \frac{q}{\gamma mc}\mathbf{v}_\perp\times\mathbf{B}\ . \tag{2}$$

Since by definition $\mathbf{v}_\perp$ is perpendicular to $\mathbf{B}$, this means that the rate of change of the direction (i.e., the frequency of rotation or gyration) is the *cyclotron frequency*

$$\omega_B = qB/(\gamma mc)\ . \tag{3}$$

Now, this is interesting. It says that for a fixed Lorentz factor (or total speed), the frequency of rotation is independent of the angle the charge makes to the magnetic field (the pitch angle). It also says that the frequency is lower when the momentum or mass or higher; you can think of this as meaning that it is more difficult for the magnetic field to bend the trajectory.

This rotation is an acceleration, with magnitude $a_\perp = \omega_B v_\perp$, so there is associated radiation. From our previous formulae, the power radiated is

$$\begin{aligned} P &= \tfrac{2q^2}{3c^3}\gamma^4 \tfrac{q^2B^2}{\gamma^2 m^2 c^2} v_\perp^2 \\ &= \tfrac{2}{3} r_0^2 c \beta_\perp^2 \gamma^2 B^2 \ . \end{aligned} \tag{4}$$

As always, we can extend this to a distribution of directions. Suppose, in particular, that we have a single speed β, but an isotropic distribution of velocities of charges. Then, averaging over directions and rewriting, we find

$$P_{\rm synch} = \frac{4}{3}\sigma_T c \beta^2 \gamma^2 U_B \ , \tag{5}$$

where U_B is the magnetic energy density $U_B = B^2/8\pi$.

Ask class: does this expression look familiar? It has *exactly* the same form as the Compton power, which was $P_{\rm compt} = \frac{4}{3}\sigma_T c\beta^2\gamma^2 U_{\rm ph}$, where $U_{\rm ph}$ was the photon energy density. Therefore, the ratio of powers is just the ratio of energy densities:

$$P_{\rm synch}/P_{\rm compt} = U_B/U_{\rm ph} \ . \tag{6}$$

Stuff like this doesn't happen by accident. There is a strong relationship between "scattering" off of field lines and scattering off of photons.

Now let's think about the spectrum of radiation produced by motion around a magnetic field. First, consider a very slowly moving particle. **Ask class:** qualitatively, how does the electric field vary? The charge is moving in a circle, so the electric field variation is sinusoidal. **Ask class:** what does this mean about the energy spectrum that we see? If we have perfectly sinusoidal motion, then a Fourier transform gives us just a single frequency. Therefore, the energy spectrum would be a single line at the cyclotron frequency ω_B. This also means, incidentally, that a slowly moving particle can only absorb or emit photons at this frequency; one consequence is that the scattering cross section goes way up for frequencies close to the cyclotron frequency.

Now suppose that the particle speed increases. **Ask class:** what happens to the electric field variation? Recall that when relativity is involved, acceleration radiation is beamed in the direction of motion of the charge. That means that instead of being perfectly sinusoidal, the electric field variation has sharper peaks. **Ask class:** what effect does that have on the Fourier transform? Since the variation isn't sinusoidal, we don't just have a single frequency any more. However, we do know that the motion is still periodic with frequency ω_B, so we can only have that frequency and multiples of it. The result is that the Fourier transform involves ω_B, $2\omega_B$, and so on, and hence the energy spectrum involves $\hbar\omega_B$, $2\hbar\omega_B$, and so on. There are therefore several lines, harmonically spaced.

Now suppose that the particle speed is highly relativistic. **Ask class:** what's the effect on the observed spectrum? In this case, the radiation is so strongly peaked forwards that many, many harmonics contribute. There are so many that the discreteness of the lines is difficult to distinguish, so that we approach a continuum spectrum. In a real situation, of course, particles of many different speeds are involved, which helps blend the spectrum if γ is not constant.

Let's go into the highly relativistic case ($\gamma \gg 1$) in more detail. We know that the radiation is beamed forward in a cone of approximate opening half-angle $1/\gamma$. We therefore might guess that since most of the radiation is emitted over an angle that is of order $2/\gamma$ of a radian, the peak frequency in the synchrotron spectrum would be of order $\gamma\omega_B$. However, this is not the case, because of the effect of light travel times.

Assume for simplicity that the pitch angle is $\alpha = 90°$, meaning that the circular motion of the particle is in the plane of our line of sight. In the cone of main emission, the particle travels a distance equal to $2/\gamma$ times the radius of curvature a of the path, $a = v/\omega_B$. Call point 1 the point in the orbit at which we are first in the beam, and point 2 the last point at which we are in the beam (thus, the particle reaches point 2 after point 1). As seen by someone on the side, the times at which these points are reached are related by $t_2 - t_1 \approx 2/(\gamma\omega_B)$. But what is seen by the observer in the beam? If $\gamma \gg 1$ then the linear distance between these points is $d \approx v(t_2 - t_1)$, where v is the speed of the particle. Here's the picture, then: at time t_1 in the frame of the center of gyration, a light pulse is sent out from the particle. At time t_2 another light pulse is sent out. But, remember, the particle is traveling with a speed close to the speed of light. Thus, by the time the second pulse is sent out, the first pulse is ahead of it, but not by much; it's only traveling a factor c/v faster, so it has only covered an extra distance of $(c/v - 1)v(t_2 - t_1) = (1 - v/c)c(t_2 - t_1)$. The observer therefore finds that the two pulses arrive separated by a much shorter time, by a factor $1 - v/c \approx 1/(2\gamma^2)$ for $\gamma \gg 1$. In general, if the pitch angle is α then the difference in arrival times is

$$\Delta t^A \approx (\gamma^3 \omega_B \sin\alpha)^{-1} . \tag{7}$$

You can see, therefore, that a little Lorentz factor goes a long way! **Ask class:** from the preceding, what do they expect to be the approximate maximum frequency observed for cyclotron radiation? It's on the order of the inverse of Δt^A; conventionally, the "critical frequency" is defined as

$$\omega_c \equiv \frac{3}{2}\gamma^3 \omega_B \sin\alpha . \tag{8}$$

Recall that ω_B scales as $1/\gamma$, so the actual dependence goes like γ^2. Now, this doesn't mean that there is *no* emission beyond ω_c, just that it drops off rapidly.

In the highly relativistic limit, the spectrum depends only on the combination $\gamma\theta$, if θ is the polar angle of the orbit. This can be used (see book) to show that in this limit the synchrotron spectrum is a universal function of ω/ω_c, which is rather convenient.

Schematically, we have

$$P(\omega) = \frac{\sqrt{3}}{2\pi} \frac{q^3 B \sin\alpha}{mc^2} F(\omega/\omega_c) \ . \tag{9}$$

The limits of F for high and low argument are

$$\begin{aligned} F(x) &\propto x^{1/3}, \qquad x \ll 1 \\ F(x) &\propto x^{1/2} e^{-x}, \qquad x \gg 1 \ . \end{aligned} \tag{10}$$

It has a peak at $x = \omega/\omega_c \approx 0.29$.

One general comment: often in astronomy spectra are characterized over limited ranges by power laws. It is therefore common to quote a *spectral index*: if we can say that for some range

$$P(\omega) \propto \omega^{-s} \ , \tag{11}$$

then s is the spectral index. Note the negative sign! A large positive spectral index means a sharply dropping spectrum. **Ask class:** What is the spectral index of the Planck function in the Rayleigh-Jeans limit? It's $s = -2$, since the power *increases* with frequency like ω^2. Another cautionary note: when you see a spectral index quoted, make sure you know if this is an *energy* spectral index or a *photon* (or number) spectral index. They differ by 1.

If the electrons have a power law distribution in energy and Lorentz factor, then if $N(E)dE = CE^{-p}dE$ the resulting power distribution is

$$P_{\rm tot}(\omega) \propto \omega^{-(p-1)/2} \tag{12}$$

implying a spectral index $s = (p-1)/2$. Note that this is the same spectral index that you get for a single Compton scattering off of electrons with a power law distribution of energies.

What about polarization? Suppose that the magnetic field direction is perpendicular to our line of sight; we'll say that the field is in the x direction and our line of sight is in the z direction. Consider a single particle, moving with Lorentz factor γ and pitch angle α. **Ask class:** qualitatively, what will the polarization be? It will be elliptically polarized in general. However, **Ask class:** if we now have a distribution of pitch angles, what happens? The right and left circular components cancel out, so we end up with linear polarization. In the midterm we considered $\alpha = \pi/2$ and found that the radiation would be 100% polarized. **Ask class:** will it be 100% polarized if a distribution of pitch angles is considered? No, because a given particle will have both x and y polarization components. The x components will cancel out in the net polarization but will contribute intensity, so the net polarization is in the y direction. It's fairly high, about 75% for frequency integrated radiation from particles of a single energy.

When we discussed bremsstrahlung we mentioned that there is an inverse process, free-free absorption. **Ask class:** what processes are similarly related to synchrotron emission? There is absorption, in which a photon is absorbed by an electron spiraling around a magnetic field. There is also stimulated emission, in which an electron emits a photon preferentially in the direction and at the frequency of a pre-existing photon. This sounds like the Einstein relations, except that we now have the strange situation of continuum states instead of discrete states, since we don't have atoms around. Thus, when we consider "states" we have to think of free particle states. This is done, qualitatively, by breaking up the continuum states into regions of phase space with phase volume h^3, then considering transitions between those. Details are in section 6.8 of Rybicki and Lightman, but I'll just quote some results.

First, **Ask class:** generically, do they expect that photons of low frequency or high frequency will have a larger optical depth to synchrotron processes going through a given region? Low frequency, since at high enough frequency one starts to run out of electrons with a large enough Lorentz factor to interact. Therefore, schematically, one expects that the low-frequency spectrum is absorbed, and this is in fact called synchrotron self-absorption. In this range, the source function is $S_\nu \propto \nu^{5/2}$; note that this is different from the Rayleigh-Jeans slope of 2, one indication that this is a nonthermal process. Note also that this is independent of the power law index for the electrons. In the optically thin high-frequency portion of the spectrum, we see the synchrotron emission directly, which has a spectral index of $(p-1)/2$ for an electron distribution with an index p. This rollover, from $\nu^{5/2}$ to $\nu^{-(p-1)/2}$, is characteristic of synchrotron radiation and is one way to identify it as the physical mechanism generating a spectrum.

Recommended Rybicki and Lightman problem: 6.1