Practice Problems Involving Computation

1. Assume you have a black hole of initial mass M_0 and some specified initial spin, and that it accretes matter at the innermost stable circular orbit. Write a computer program to calculate the dimensionless spin parameter $j = a/M = J/M^2$ of the hole as a function of its current mass M. In this problem, as is common in general relativity, we use units in which $G = c = 1$.

(a) If it starts with $j = 0$ and always accretes matter in the prograde direction, then to within 0.1% how much mass does it accrete to get to j=0.5 and 0.9? What is the maximum j reached, to an accuracy of 10^{-4}?

(b) Suppose you start with a near-extremal Kerr hole ($j = 0.999$) and accrete matter in *retrograde* orbits at the innermost stable circular orbit. To within 0.1%, how much mass does it accrete to get to j=0.99, 0.9, 0.5, and 0?

For this problem you need the following formulae. The specific angular momentum (i.e., per unit rest mass) of a particle in a circular geodesic at radius r around a black hole of mass M and spin parameter $a = jM$ is

$$u_\phi = \pm \frac{\sqrt{Mr}\left(r^2 \mp 2a\sqrt{Mr} + a^2\right)}{r\left(r^2 - 3Mr \pm 2a\sqrt{Mr}\right)^{1/2}} \ . \tag{1}$$

Here the upper sign is for prograde orbits and the lower sign is for retrograde orbits. If a mass m is accreted, then, the angular momentum changes by mu_ϕ. The radius of the innermost stable circular orbit is

$$r_{\rm ISCO} = M\left\{3 + Z_2 \mp [(3 - Z_1)(3 + Z_1 + 2Z_2)]^{1/2}\right\} \ , \tag{2}$$

where again the upper sign is for prograde and the lower is for retrograde. Here we use

$$Z_1 = 1 + \left(1 - j^2\right)^{1/3}\left[(1 + j)^{1/3} + (1 - j)^{1/3}\right] \tag{3}$$

and

$$Z_2 = \left(3j^2 + Z_1^2\right)^{1/2} \ . \tag{4}$$

2. (from the in-preparation new version of Hansen and Kawaler) Plot the total opacity $\kappa_{\rm tot}$ for densities of 10 and 10^4 g cm^{-3}, including three types of radiative opacity (bound-free, free-free, and H$^-$). Assume cosmic composition ($X = 0.7$, $Y = 0.28$, $Z = 0.02$) and use the

following equations for free-free, bound-free, and H^- opacity (all opacities are in cm^2 g^{-1}):

$$\begin{aligned}
\kappa_{f-f} &\approx 10^{23}\rho Z_c^2 T^{-3.5} \\
\kappa_{b-f} &\approx 4\times 10^{25} Z(1+X)\rho T^{-3.5} \\
\kappa_{H^-} &\approx 2.5\times 10^{-31}(Z/0.02)\rho^{1/2}T^9
\end{aligned} \tag{5}$$

Here Z_c is the average nuclear charge. Plot for a temperature range of $10^5 - 10^8$ K. Now include conduction using

$$\kappa_{\rm cond} \approx 4\times 10^{-8} Z_c^2 (T/\rho)^2 \tag{6}$$

and replot. How much of a change is there? Remember to combine the various forms of opacity correctly!

3. The following is basically Problem 3 in chapter 6 of Hansen and Kawaler, "Stellar Interiors".

Suppose you have a gram of pure helium (as ^{4}He) in the center of a pre-helium flash red supergiant. The density and temperature of the gram are, respectively, $\rho = 2\times 10^5$ g cm^{-3} and $T = 1.5\times 10^8$ K. This is hot enough to burn helium by the triple-α reaction—which is the only reaction you will use. The energy generation rate for the reaction is given by

$$\epsilon_{3\alpha} = \frac{5.1\times 10^8 \rho^2 Y^3}{T_9^3} e^{-4.4027/T_9}\ \text{erg g}^{-1}\ \text{s}^{-1}\ . \tag{7}$$

Here ρ is in g cm^{-3}, Y is the fractional abundance of helium by number, and $T_9 = T/10^9$ K.

You are now to follow the time evolution of the gram as helium burning proceeds by computing the temperature, $T(t)$, as a function of time. Start the clock running at time zero at the conditions stated. Assume that the density remains constant for all time, that no heat is allowed to leave the gram, and find $T(t)$ until that time when the material begins to become nondegenerate. For this use the nonrelativistic demarcation line $\rho \approx 10^{-8}T^{3/2}$; densities above this are degenerate, below that are not.

For a given input of heat energy, calculate the change in temperature using the specific heat c_V, where $\Delta T = (\Delta E/\Delta m)/c_V$; that is, the change in temperature of some mass Δm is equal to the energy input per gram input into that mass $(\Delta E/\Delta m)$ divided by the specific heat. The specific heat is the sum of the electron specific heat

$$c_{Ve} = \frac{1.35\times 10^5}{\rho} T x_f (1+x_f^2)^{1/2}\ \text{erg g}^{-1}\ \text{K}^{-1} \tag{8}$$

where $\rho \approx 2\times 10^6 x_f^3$ allows solution for x_f, and the ideal gas specific heat for helium

$$c_{VHe} = 3k/4m_p \tag{9}$$

so that $c_V = c_{Ve} + c_{VHe}$.

So the problem doesn't become too difficult, assume that the helium concentration does not change with time.

Produce a plot of the temperature versus time. As a sanity check, indicate whether the answer you get is reasonable, and what criteria you use to make that determination. If your answer is not reasonable, tell me what it should be and why. At what temperature does the matter become nondegenerate? What will happen to the density and temperature of the matter in the core when nondegeneracy hits (don't calculate quantitatively, just answer qualitatively)?

4. For a polytropic equation of state (in which $P(r) = K\rho^{1+1/n}(r)$, where K is some constant, P is the pressure, and ρ is the density), the equations of stellar structure are often simplified and rewritten to produce the *Lane-Emden equation*

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right) = -\theta^n \ . \tag{10}$$

Here θ is a renormalized density, where $\rho(r) = \rho_c\theta^n(r)$ for a central density ρ_c, and ξ is a dimensionless radial coordinate defined by $r = r_n\xi$, where the scale length r_n is

$$r_n^2 = \frac{(n+1)P_c}{4\pi G\rho_c^2} \ . \tag{11}$$

The scalings don't really matter; to solve for the structure of a star under these assumptions, you pick an index n and then solve for $\theta_n(\xi)$. Also of interest is the scaled radius of the surface, ξ_1, which is the first ξ at which $\theta = 0$. There are three analytic solutions:

$$\begin{array}{lll} n=0 & \theta_0(\xi) = 1 - \xi^2/6 & \xi_1 = \sqrt{6} \\ n=1 & \theta_1(\xi) = \sin\xi/\xi & \xi_1 = \pi \\ n=5 & \theta_5(\xi) = [1+\xi^2/3]^{-1/2} & \xi_1 \to \infty \ . \end{array} \tag{12}$$

Your task is to write a program to solve the Lane-Emden equation numerically.
(a) Check your results against the analytic solutions for $n = 0$, $n = 1$, and $n = 5$. Do this by plotting θ versus ξ for the analytical and for your numerical solution on the same graph (one graph per value of n) and by listing the value of ξ_1.

(b) What is ξ_1 for $n = 3$ and $n = 3/2$, the two most interesting values in practice?

(c) Solve the Lane-Emden equation for the Sun, assuming it is an ideal gas ($n = 3/2$). What is the central density, if the mass and radius are fixed at their actual values?

Hint: try the simplest numerical integrations first! I won't demand ultraprecise answers; three significant figure agreement with analytic solutions is fine.