

Mention: programming, probably second homework

Initial question: Suppose that my boy Lord Voldemort waved his wand and stopped all fusion in the Sun. How long would it take before we noticed the Sun's light dimming?

Radiation forces

Before heading to scattering, one comment about radiation forces. The book has a discussion on it (end of §1.4), but I think it's unnecessarily complicated (and it's not general, either). The easy way is to remember that force is change in momentum per time, $\mathbf{F} = d\mathbf{p}/dt$. The magnitude of the momentum of a photon is $p = E/c$, where $E = \hbar\omega$ is the photon's energy. Therefore, to figure out the net force on a particle that is absorbing or scattering radiation, just (1) compute the incident rate of momentum on the particle, (2) compute the rate of momentum after scattering, and (3) take the difference.

Scattering and Random Walks

We're now going to take the first steps towards understanding how radiation interacts with matter. In this lecture we are going to spend substantial time on an idealization: coherent, frequency-independent, isotropic scattering, along with random walks. There are two reasons for this. First, it allows us to define a number of important quantities and explore concepts in a simple setting, which will provide a basis for more complicated processes. Second, the concepts of random walks and diffusion are easy to understand but are also far more general than they appear, so an understanding of them allows order of magnitude estimates in a rather broad array of problems.

So, let's set up the physical situation: a photon travels along merrily, and at some point it hits a particle and bounces off. In general the nature of the scattering could depend on lots of things, but for now we'll assume the scattering is coherent (the photon energy does not change), frequency-independent (the rate or cross section of scatterings does not depend on the photon's frequency), and isotropic (the probability of the photon scattering into a particular direction is independent of the direction). That's just to simplify things. Now we need to define several quantities: cross section, opacity, mean free path, and optical depth.

Cross sections.—**Ask class:** in everyday life, how would you define cross section? Often, simply its geometrical area. If you throw projectiles at me, the ones that hit will hit in my area in profile, maybe 0.6 m^2 face-on, and half that edge-on. As long as the projectile is small compared to me, its nature doesn't matter much; a water balloon or a rotten tomato will have the same effective cross section of interaction. **Ask class:** can they think of another way to define it? Suppose there were to be a steady pelting by rotten tomatoes, but that everyone had such bad aim that any area had the same probability of being hit as any other area. We can then define the tomato flux, F_{tom} , as the number per area per second. My cross

section is then $R_{\text{tom}}/F_{\text{tom}}$, where R_{tom} is the rate at which they hit me in number per second. In the microscopic world, it is this latter definition that is more general and therefore more useful.

The cross section is not always independent of the nature of the projectiles. **Ask class:** if the “projectiles” are photons, can they think of an everyday example of this? A pane of glass is nearly transparent to visible light, but is opaque to infrared, hence the greenhouse effect. The effective cross section is defined as rate divided by flux, again, making the effective cross section much less for visible light than for infrared. Similarly, for some reaction $X + \alpha \rightarrow Y + \beta$, we define a cross section $\sigma_{\alpha\beta}(v)$ as the number of such reactions per second per target divided by the incident flux of the projectiles. Then the reaction rate per volume is $\sigma_{\alpha\beta}(v)vn_{\alpha}n_X \text{ cm}^{-3} \text{ s}^{-1}$. If the target and projectile are the same, have to divide by two to avoid double-counting. **Ask class:** can they think about how to get the average rate if there is a distribution of particle velocities? Have to integrate $\sigma_{\alpha\beta}v$ over that distribution.

Opacity.—We can think of the cross section as the “effective area” of a single particle to a particular interaction (scattering in our case). We can also define the opacity, which is the total effective area per mass: κ is in $\text{cm}^2 \text{ g}^{-1}$. So, for a gram of material, the “effective area” to the given interaction is $\kappa \text{ cm}^2$. You might, therefore, be tempted to calculate opacity by simply taking the total number of particles per gram and multiplying by the cross section per particle. **Ask class:** why is this wrong? Be careful! The particles that count are the ones that are involved in the particular interaction, and may be a small fraction of the total number. Example: the cross section to scattering of low-energy photons off of free electrons is the Thomson cross section $\sigma_T = 6.65 \times 10^{-25} \text{ cm}^2$. However, if material is neutral (and there are therefore no free electrons), the *opacity* to Thomson scattering is zero! I’ll be obnoxiously reiterative about this, since the distinction between opacity and cross section is important conceptually.

Mean free path.—This is the “average” distance traveled by a photon between interactions. For example, if the number density of scatterers is $n \text{ cm}^{-3}$ and the cross section is $\sigma \text{ cm}^2$, then the mean free path (mfp) is $l = 1/(n\sigma)$. Similarly, if the *total* mass density is $\rho \text{ g cm}^{-3}$ and the opacity to the scattering process is $\kappa \text{ cm}^2 \text{ g}^{-1}$ then $l = 1/(\rho\kappa)$. Note that if you forget these expressions you can get them from dimensional arguments. If the scatterers are distributed randomly (e.g., not in a lattice) then the probability that a photon will travel a distance d or greater before scattering is $\exp(-d/l)$, from the Poisson distribution. You can verify that this distribution gives a mean distance of l . **Ask class:** can they give me an example of how a non-random distribution can mess this up? Imagine a perfect cubic lattice. In some directions, one can travel forever without hitting anything. We’ll almost always deal with gasses or fluids, though, where the assumption of random distribution is good.

Optical depth.—If we imagine a path in a medium, parameterized by the variable s , then the optical depth τ is

$$\tau = \int \frac{ds}{l(s)}, \quad (1)$$

where $l(s)$ is the mean free path along each point. In words, if you are forced to travel straight in some direction (that is, you aren't deflected by scatterings), the optical depth is the number of mean free paths you have traveled. For example, if the distance is one mean free path, the optical depth is 1. If it is two mean free paths, the optical depth is 2, and so on. Note that the mean free path can change along the way (e.g., if the density or ionization state changes). A medium with $\tau \ll 1$ is called optically thin, and it means that the average photon does not interact. A medium with $\tau \gg 1$ is called optically thick, and it means that the average photon interacts many times (much more than τ in this case; actually, it's more like τ^2 [see below]). The concept of optical depth is extremely useful, not least because in a sense this is what the photon “sees”: you can think of the photon wandering around in optical depth space rather than in real space, which can simplify the treatment of many problems.

Now let's think about what happens to the location of the photon when it undergoes multiple scatterings.

Diffusion and random walks: It would be wonderful to be able to treat the motion of photons as if it were a random walk, independent of frequency. Of course, a problem with this is that photons of different energies have different opacities, so it isn't completely straightforward. For the moment, however, let's ignore this and assume that the opacity is independent of the frequency (“gray atmosphere” approximation).

If the photons were to simply scatter, then their progress would be a random walk. Suppose that in step 1 the photon travels a distance and direction \mathbf{r}_1 , in step 2 \mathbf{r}_2 , and so on, in random directions but with the same distribution of distances. The mean location after N scatterings is zero (it has to be, due to the isotropy), but the mean square distance is

$$\langle \mathbf{R}^2 \rangle = \langle \mathbf{r}_1^2 \rangle + \langle \mathbf{r}_2^2 \rangle + \langle \mathbf{r}_3^2 \rangle + \dots + 2\langle \mathbf{r}_1 \cdot \mathbf{r}_2 \rangle + 2\langle \mathbf{r}_1 \cdot \mathbf{r}_3 \rangle + \dots \quad (2)$$

The cross terms all average to zero, but the squared terms are all the same and are the mean square distance traveled in a single scatter. The net result is that the mean square distance after N steps is Nl^2 , where l^2 is the mean square distance for a single scattering. Therefore, we get the result that the average distance from the origin is $\sqrt{N}l$. **Ask class:** how does this depend on the number of dimensions in which the random walk is performed? For example, does it matter if we do this in one dimension or three? It is basically independent; the coefficient may change somewhat, but the dependence on $N^{1/2}$ doesn't. This is an example of how general random walks are. Similarly, if the optical depth to escape is $\tau \gg 1$, the photon typically undergoes $\sim \tau^2$ scatterings before it escapes. If the medium is optically

thin, $\tau \ll 1$, of order τ scatterings on average are required. For most rough estimates you can use just $\max(\tau, \tau^2)$ for the number of scatterings. In stellar interiors, $\tau \gg 1$ so we're in the random walk limit and can treat the process of radiation transfer as diffusion. In fact, this leads to some rather dramatic results, so let's try an example.

Suppose that photons in the Sun travel by diffusion, that the Sun has a uniform density and is completely ionized, and that the photon cross section is always equal to the Thomson cross section, **Ask class:** approximately how long would it take photons to diffuse from the center of the Sun to the surface? I'll provide necessary numbers for opacities, etc.

Answer:

To a good approximation, photons will diffuse in a random walk. To within factors of ~ 2 , the number of steps taken in a random walk before escape if the total optical depth to the Sun's radius R is τ_R is about τ_R^2 . The mean free path in each scattering is $l = R/\tau_R$, so the total distance traveled is $\sim \tau_R R$.

The average density of the Sun is about 1.4 g cm^{-3} , which would be about 10^{24} particles per cubic centimeter if the Sun were pure hydrogen; the 25% helium by mass is a relatively minor correction. The Thomson cross section is $\sigma_T = 6.65 \times 10^{-25} \text{ cm}^2$, and the radius of the sun is $R \approx 7 \times 10^{10} \text{ cm}$, so the optical depth is $\tau_R = n\sigma R \approx 10^{24} \times 6.65 \times 10^{-25} \times 7 \times 10^{10} \approx 5 \times 10^{10}$. The distance traveled by a photon is therefore $\tau_R R \approx 3 \times 10^{21} \text{ cm}$, which would take light $\sim 10^{11} \text{ s}$ or about 3,000 years to travel. The actual time would be much greater than this (millions of years), because in most of the Sun other forms of opacity are much more important, so the optical depth is a lot higher. But this simple calculation shows that you'd have to wait a long time before the radiation leaked out!

Generality of random walks

As promised, in diffusion situations (high optical depth) the random walk idea works in a whole host of problems. A prime reason is the central limit theorem. Suppose that we have an experiment with a certain probability distribution of results. Assume that the distribution is non-pathological (which, loosely speaking, means that the differential distribution doesn't go to infinity anywhere in the way that, e.g., a delta function does, and that the standard deviation isn't infinite). Now suppose that we perform this experiment a large number of times and average the results to get a single number. If *that* procedure is repeated many times, we get a distribution of the averages of many experiments. The central limit theorem states that the distribution of the averages tends to a Gaussian, with the mean of the Gaussian equal to the mean of the original distribution and the standard deviation equal to σ_0/\sqrt{N} , where σ_0 is the standard deviation of the original distribution and N is the number of experiments.

Ask class: How does that relate to random walks? The mean change in location is zero for an unbiased random walk, but the standard deviation is of order the mean free path. After N steps the mean location is still the original location, but the standard deviation is N times the standard deviation of the average, or $\sim Nl/\sqrt{N} \sim l\sqrt{N}$. Thus, other than some usually insignificant factors of order unity, the main conclusions of the random walk treatment are *entirely independent* of the detailed distribution of the step sizes. It could be Poisson, it could be Gaussian, it could be some kind of bimodal doohicky, but the result is the same. In many astronomical problems you only need an answer to an order of magnitude, so guessing that there will be $\sim \max(\tau, \tau^2)$ scatterings is fine. For example, we'll see that in reality scattering of a photon off an electron is not isotropic. This, however, introduces only minor factors and for many applications one can treat the random walk as isotropic scattering.

Note that in this discussion we are thinking about two distinct types of standard deviation. One is the standard deviation in the distance traveled in a *single step*, and the other is the standard deviation of the *location* after many steps. The single step standard deviation clearly depends on the distribution of distances. However, what we claim here is that after many steps, the standard deviation of the location depends only on the mean distance traveled per step rather than on the details of the distribution of the distance in a given step. This is where the central limit theorem enters: because the standard deviation of the average of many draws N_{draw} from a distribution decreases like $N_{\text{draw}}^{-1/2}$, but the typical distance from the origin scales like $\ell N_{\text{draw}}^{1/2}$, the details do not matter after enough steps.

Other types of random walks

So far we've made two assumptions about our random walk: (1) the directions of scatters are distributed randomly and uniformly, and (2) the distribution of distances traveled does not depend on direction. Let's relax the second assumption. In particular, let's assume that the particle (or photon or quantity of interest) can take bigger steps towards the origin than away from it. This is called a *damped random walk*.

To treat this, let's revisit the original problem but treat it in a way that can be generalized. Consider for simplicity a one-dimensional random walk in which all steps are of unit length, with a probability of 0.5 for a step to the right and 0.5 for a step to the left. Suppose that the particle is currently at location $N^{1/2}$. To get the root mean square distance after the next step, we notice that the average square distance after that step is $0.5(N^{1/2} - 1)^2 + 0.5(N^{1/2} + 1)^2 = N + 1$, so the root mean square distance is $(N + 1)^{1/2}$. Sure enough, this is consistent with the idea that after N steps the typical distance is $N^{1/2}$ times the mean free path for an undamped random walk. Note in particular that the distance from the origin grows without bound in this case.

For a damped random walk it's different. In many cases, the damping means that

although the distance from the origin may increase at first, it gets to a point where another step typically doesn't change the root mean square distance from the origin. In that case, the distance saturates; it may still wander around, but won't get arbitrarily far from the origin. Let's consider an example. Suppose that we have a one-dimensional random walk with unit length steps, but that at a distance $d > 1$ from the origin the probability of a step towards the origin is $(d - 1)/d$ whereas the probability of a step away is $1/d$. The mean square distance is then

$$\langle D^2 \rangle = (1/d)(d + 1)^2 + [(d - 1)/d](d - 1)^2 . \quad (3)$$

We can look for a fixed point by setting $\langle D^2 \rangle = d^2$, and we find $d = 5/2$. That means that d will tend towards $5/2$ and will not grow without bound.

Damped random walks have many applications in astrophysics, and it's important to remember that the quantity undergoing the random walk does not have to be a distance. For example, consider a dust grain in a protoplanetary disk being buffeted by supersonic turbulence. Here, it's the velocity of the dust relative to the average motion of the fluid that matters: if the velocity gets to be too high, then drag tends to lower it. Another example is the angular momentum of a black hole that grows by accreting chunks of matter that fly in at random angles. The angular momentum of the chunk at accretion is larger if the orbit is retrograde relative to the spin of the black hole than if it is prograde, so the spin of the black hole tends to be decreased by this process.