Binary Sources of Gravitational Radiation

We now turn our attention to binary systems. These obviously have a large and varying quadrupole moment, and have the additional advantage that we actually know that gravitational radiation is emitted from them in the expected quantities (based on observations of double neutron star binaries). The characteristics of the gravitational waves from binaries, and what we could learn from them, depend on the nature of the objects in those binaries. We will therefore start with some general concepts and then discuss individual types of binaries.

First, let's get an idea of the frequency range available for a given type of binary. There is obviously no practical lower frequency limit (just increase the semimajor axis as much as you want), but there is a strict upper limit. The two objects in the binary clearly won't produce a signal higher than the frequency at which they touch. If we consider an object of mass M and radius R, the orbital frequency at its surface is $\sim \sqrt{GM/R^3}$. Noting that $M/R^3 \sim \rho$, the density, we can say that the maximum frequency involving an object of density ρ is $f_{\rm max} \sim (G\rho)^{1/2}$. This is actually more general than just orbital frequencies. For example, a gravitationally bound object can't rotate faster than that, because it would fly apart. In addition, you can convince yourself that the frequency of a sound wave through the object can't be greater than $\sim (G\rho)^{1/2}$. Therefore, this is a general upper bound on dynamical frequencies.

This tells us, therefore, that binaries involving main sequence stars can't have frequencies greater than $\sim 10^{-3}-10^{-6}$ Hz, depending on mass, that binaries involving white dwarfs can't have frequencies greater than $\sim 0.1-10$ Hz, also depending on mass, that for neutron stars the upper limit is $\sim 1000-2000$ Hz, and that for black holes the limit depends inversely on mass (and also spin and orientation of the binary). In particular, for black holes the maximum imaginable frequency is on the order of $10^4(M_\odot/M)$ Hz at the event horizon, but in reality the orbit becomes unstable at lower frequencies (more on that later).

Now suppose that the binary is well-separated, so that the components can be treated as points and we only need take the lowest order contributions to gravitational radiation. Temporarily restricting our attention to circular binaries, how will their frequency and amplitude evolve with time?

Let the masses be m_1 and m_2, and the orbital separation be R. We argued in the previous lecture that the amplitude a distance $r \gg R$ from this source is $h \sim (\mu/r)(M/R)$, where $M \equiv m_1 + m_2$ is the total mass and $\mu \equiv m_1 m_2/M$ is the reduced mass. We can rewrite the amplitude using $f \sim (M/R^3)^{1/2}$, to read

$$\begin{aligned} h &\sim \mu M^{2/3} f^{2/3}/r \\ &\sim M_{ch}^{5/3} f^{2/3}/r \end{aligned} \tag{1}$$

where M_{ch} is the "chirp mass", defined by $M_{ch}^{5/3} = \mu M^{2/3}$. The chirp mass is named that because it is this combination of μ and M that determines how fast the binary sweeps, or chirps, through a frequency band. When the constants are put in, the dimensionless gravitational wave strain amplitude (i.e., the fractional amount by which a separation changes as a wave goes by) measured a distance r from a circular binary of masses M and m with a binary orbital frequency $f_{\rm bin}$ is (Schutz 1997)

$$h = 2(4\pi)^{1/3} \frac{G^{5/3}}{c^4} f_{\rm GW}^{2/3} M_{ch}^{5/3} \frac{1}{r} \ , \tag{2}$$

where $f_{\rm GW}$ is the gravitational wave frequency. Redshifts have not been included in this formula.

The luminosity in gravitational radiation is then

$$\begin{aligned} L &\sim 4\pi r^2 f^2 h^2 \\ &\sim M_{ch}^{10/3} f^{10/3} \\ &\sim \mu^2 M^3/R^5 \ . \end{aligned} \tag{3}$$

The total energy of a circular binary of radius R is $E_{\rm tot} = -G\mu M/(2R)$, so we have

$$\begin{aligned} dE/dt &\sim \mu^2 M^3/R^5 \\ \mu M/(2R^2)(dR/dt) &\sim \mu^2 M^3/R^5 \\ dR/dt &\sim \mu M^2/R^3 \ . \end{aligned} \tag{4}$$

What if the binary orbit is eccentric? The formulae are then more complicated, because one must then average properly over the orbit. This was done first to lowest order by Peters and Matthews (1963) and Peters (1964), by calculating the energy and angular momentum radiated at lowest (quadrupolar) order, and determining the change in orbital elements that would occur if the binary completed a full Keplerian ellipse in its orbit. That is, they assumed that to lowest order, they could have the binary move as if it experienced only Newtonian gravity, and integrate the losses along that path.

Before quoting the results, we can understand one qualitative aspect of the radiation when the orbits are elliptical. From our derivation for circular orbits, we see that the radiation is emitted much more strongly when the separation is small, because $L \sim R^{-5}$. Consider what this would mean for a very eccentric orbit $(1-e) \ll 1$. Most of the radiation would be emitted at pericenter, hence this would have the character of an impulsive force. With such a force, the orbit will return to where the impulse was imparted. That means that the pericenter distance would remain roughly constant, while the energy losses decreased the apocenter distance. As a consequence, the eccentricity decreases. In general, gravitational radiation will decrease the eccentricity of an orbit.

The Peters formulae bear this out. If the orbit has semimajor axis a and eccentricity e, their lowest-order rates of change are

$$\langle\frac{da}{dt}\rangle = -\frac{64}{5}\frac{G^3\mu M^2}{c^5a^3(1-e^2)^{7/2}}\left(1+\frac{73}{24}e^2+\frac{37}{96}e^4\right) \tag{5}$$

and

$$\langle\frac{de}{dt}\rangle = -\frac{304}{15}e\frac{G^3\mu M^2}{c^5a^4(1-e^2)^{5/2}}\left(1+\frac{121}{304}e^2\right) \tag{6}$$

where the angle brackets indicate an average over an orbit. One can show that these rates imply that the quantity

$$ae^{-12/19}(1-e^2)\left(1+\frac{121}{304}e^2\right)^{-870/2299} \tag{7}$$

is constant throughout the inspiral.

Do we have evidence that these formulae actually work? Yes! Nature has been kind enough to provide us with the perfect test sources: binary neutron stars. Several such systems are known, all of which have binary separations orders of magnitude greater than the size of a neutron star, so the lowest order formulae should work. Indeed, the da/dt predictions have been verified to better than 0.1% in a few cases. The de/dt predictions will be much tougher to verify, though. The reason for the difference is that de/dt has to be measured by determining the eccentricity orbit by orbit, whereas da/dt has a manifestation in the total phase of the binary, so it accumulates quadratically with time. These systems provide really spectacular verification of general relativity in weak gravity. In particular, in late 2003 a double pulsar system was detected, that in addition has the shortest expected time to merger of any known system (only about 80 million years). Having two pulsars means that extra quantities can be measured (such as the relative motion, which gives us the mass ratio), and in fact the system is now dramatically overconstrained (more things measured than there are parameters in the theory). The tests of GR by observations of binary neutron star systems deservedly resulted in the 1993 Nobel Prize in physics going to Hulse and Taylor, who discovered the first such binary.

We are therefore quite confident that, at least in weak gravity, gravitational radiation exists as advertised. What happens in strong gravity?

When two masses are close enough to each other, the Peters formulae do not quite describe their motion. Instead, there are additional terms corresponding to higher order moments of the mass and current distributions: the octupole, hexadecapole, and so on. This is often expressed in terms of equations of motion that include the Newtonian acceleration and a series of "post-Newtonian" (PN) terms. The order of a term is labeled by the number of factors of M/r by which it differs from Newtonian: for example, the 1PN term is proportional to M/r times the Newtonian acceleration. Since $v^2 \sim M/r$ in a binary orbit, there can also be half-power terms. The first several corrections are at the 1PN, 2PN, 2.5PN (this is where gravitational radiation losses first enter), 3PN, and 3.5 PN orders.

The equations of motion have been fully, rigorously established up to 3PN order, but the algebra is daunting and serious technical challenges exist that make it difficult to determine unambiguously the coefficients in each succeeding set of terms. We note that, fortunately, tidal effects only enter at the 5PN order, which one can justify by realizing that tidal couples have a $1/r^6$ energy dependence, or five powers of r greater than the Newtonian potential. Therefore, for many purposes, tidal effects can be neglected. The post-Newtonian approach is useful, but problematic because succeeding terms are not much smaller than the terms before them. Another way to put this is that the Newtonian acceleration is overwhelmingly dominant for an extremely wide range of separations (out to infinity, in fact), but the range in which the 1PN term is necessary but the 2PN term is negligible is small, and this becomes even more true for the higher order terms. One can therefore often make good progress by taking the lowest-order term, and since the 2.5PN term is the lowest-order that involves energy and angular momentum loss, one can use the Newtonian plus 2.5PN terms. However, more terms turn out to be necessary to get sufficiently accurate waveforms for analysis of future gravitational wave data streams.

Various clever attempts have been made to recast the expansions into forms that converge faster than Taylor series. For example, a path adopted by Damour and Buonanno is to pursue equivalent one-body spacetimes in which an effective test particle moves, and to then graft on the effects of gravitational radiation losses. They also use Padé resummation, in which the terms are ratios of polynomials, in the hopes that this can more naturally model the singularity of black hole spacetimes. It is not obvious that in practice this confers special advantages.

For example, the "pit in the potential" aspect of general relativity, in which at strong distances the effective gravitational acceleration is larger than it would be in Newtonian gravity, implies that if one considers circular orbits of smaller and smaller radii, there will be an orbital separation that minimizes the specific angular momentum of the system. This is in strong contrast with Newtonian gravity, for which the specific angular momentum decreases monotonically with decreasing radius. The existence of a minimum in the specific angular momentum implies that inside that radius, circular orbits are unstable, hence this determines the innermost stable circular orbit (ISCO). If a binary spirals inside the ISCO it enters a plunge phase. For a test particle moving under the influence of gravity alone the ISCO is a precisely determined radius, but finite losses of angular momentum (e.g., from magnetic effects or gravitational radiation) blur the line. For two equal-mass nonspinning objects, one can compute the separation and angular momentum at the ISCO to different PN orders, and although the various approaches agree (e.g., Taylor-like PN expansions or Padé resummation in the equivalent one-body approach), the corrections from, e.g., 2PN to 3PN order are disturbingly large.

One interesting effect that emerges from the higher-order studies of binary inspirals is that gravitational radiation carries away net linear momentum, hence the center of mass of the system moves in an ever-widening spiral. We can understand this as follows (following an idea of Alan Wiseman). In an unequal-mass binary, the lower-mass object moves faster. As the speed in orbit becomes relativistic, the gravitational radiation from each object becomes beamed, with the lower-mass object producing more beaming because it moves faster. Therefore, at any given instant, there is a net kick against the direction of motion of the lower-mass object. If the binary were forced to move in a perfect circle, the center of mass of the system would simply go in a circle as well. However, because in reality the orbit is a tight and diminishing spiral, the recoil becomes stronger with time and the center of mass moves in an expanding spiral. Note that by symmetry, equal-mass nonspinning black holes can never produce a linear momentum kick, and that if the mass ratio is gigantic the fractional energy release is small and therefore so is the kick. For nonspinning holes, the optimal ratio for a kick is about 2.6.

This process is potentially important astrophysically because if the final merged remnant of a black hole inspiral is moving very rapidly, it could be kicked out of its host stellar system, with possibly interesting implications for supermassive black holes and hierarchical merging. There have therefore been a number of calculations of the expected kick. It has turned out that these are very challenging. The primary reason is that most of the action is near the end, when the black holes are close to each other and simple approximations to the orbit are inaccurate. Analytic calculations (recent examples include Favata, Hughes, and Holz 2004; Blanchet, Qusailah, and Will 2005; Damour and Gopakumar 2006), suggest that the kick due to inspiral from infinity to the ISCO is minimal, but that the final plunge could produce interesting speeds. In the last year there has been tremendous progress in numerical relativity, and this has allowed an estimate of about 100 km/s for a 1.5:1 mass ratio of nonspinning black holes (Baker et al. 2006) as well as lower limits to the kick for a few other mass ratios (Herrmann, Shoemaker, and Laguna 2006).

Generically, if two black holes coalesce, how does it happen? In this field it is standard (and reasonable) to divide the whole process up into three stages. The first stage is inspiral, which follows the binary from large separations to when the binary has reached the stage of dynamical instability. That is, inspiral is roughly where the binary is outside the innermost stable circular orbit, so the motion is mostly azimuthal. Inside the ISCO, the motion becomes a plunge, and this happens on a dynamical time scale. As the event horizons disturb each other and finally overlap, the spacetime becomes extremely complicated and must be treated numerically. This is called the merger phase.

Ultimately, of course, the "no hair" theorem guarantees that the system must settle into a Kerr spacetime. It does this by radiating away its bumpiness as a set of quasinormal modes. The lowest-order, and longest-lived, of the modes is the $l = 2,\ m = 2$ mode. When all but this mode have essentially died away, the system has entered the period of ringdown. With only a single mode left, the ringdown phase can be treated numerically. The result is that the frequency f_{qnr} of the gravitational radiation, as well as the quality factor $Q \equiv \pi f_{qnr}\tau$ (where τ is the characteristic duration of the mode; this measures how many cycles the ringing lasts) depend on the effective spin $j \equiv cJ/GM^2$ of the final black hole (sometimes $\hat{a}$ is used instead of j). Echeverria (1989) gives fitting formulae valid to $\sim 5\%$:

$$\begin{array}{ll} f_{qnr} & \approx [1 - 0.63(1-j)^{0.3}](2\pi M)^{-1} \\ Q & \approx 2(1-j)^{-0.45}\ . \end{array} \tag{8}$$

Thus more rapidly spinning remnants have higher frequencies and last for more cycles. This could allow identification of the spin based on the character of the ringdown.

We can make rough estimates of the energy released in each phase as a function of the reduced mass μ and total mass M of the system. Since the inspiral phase goes from infinity to the ISCO, the energy released is simply μ times the specific binding energy at the ISCO, so $E_{\rm inspiral} \sim \mu$. What about the merger and ringdown phases? We know that the strain amplitude is $h \sim (\mu/r)(M/R)$, where r is the distance to the observer and R is the dimension of the system. For the merger and ringdown phases, $R \sim M$, so $h \sim \mu/r$. We also know that the luminosity is $L \sim r^2 h^2 f^2$, so $L \sim \mu^2 f^2$, and if the phase lasts a time τ then the total energy released is $E \sim \mu^2 f^2 \tau$. But the characteristic frequency is $f \sim 1/M$ and the characteristic time is $\tau \sim M$, so we have finally $E \sim \mu^2/M$, or a factor $\sim \mu/M$ times the energy released in the inspiral. The exact values for a particular mass ratio are somewhat in dispute, but for an equal-mass nonspinning black hole binary, $E_{\rm inspiral} \sim 0.06M$ and $E_{\rm merger}$ and $E_{\rm ringdown}$ are probably $\sim 0.01M$. Note that for highly unequal mass binaries ($\mu \ll M$), the inspiral produces much greater total energy than the merger or ringdown. This is one reason why analyses of extreme mass ratio inspirals have ignored the merger and ringdown phases (see later).