Stochastic Backgrounds

For our last lecture, we will focus on stochastic backgrounds, with an emphasis on primordial gravitational waves. To get a handle on these issues, we need to think in terms of broad bands of frequency with many sources, rather than the signal produced by an individual source. For this purpose, it is useful to reproduce a discussion due to Sterl Phinney about the relation between gravitational wave spectral density from a class of sources, and the total energy released in those sources over their lifetimes.

Before moving into the details, we can state the general idea very simply. Suppose we have a closed box, and sources in the box that emit some total average energy $\langle E \rangle$ during their lifetimes. If the number density of the sources is n, then the energy density in the box is $n\langle E \rangle$. If the energy emitted between frequencies f and $f+df$ is $(dE/df)df$, and if there is no redshifting, then the total energy density between f and $f+df$ is $n(dE/df)df$. Expansion of the universe changes these expressions, as we will now explore.

Following Phinney's derivation, let the frequency of a gravitational wave in the rest frame of a source be f_r, and the observed frequency be $f = f_r/(1+z)$. Let the emitted energy between f_r and $f_r + df_r$ be $(dE/df_r)df_r$. This is the total energy emitted over the lifetime of the source, in all directions, and is measured in the rest frame of the source. Let us also define $\Omega_{\rm GW}(f)$ as follows. Suppose that $d\rho_{\rm GW}(f)c^2/df$ is the present-day energy density in gravitational waves of frequencies between f and $f + df$. Let $\rho_c c^2$ be the mass-energy density needed to close the universe. Then

$$\Omega_{\rm GW} \equiv \frac{1}{\rho_c c^2}\frac{d\rho_{\rm GW}(f)c^2}{d\ln f}\ . \tag{1}$$

That is, $\Omega_{\rm GW}$ is the ratio of the present-day energy density in gravitational waves in a logarithmic interval around f to the critical energy density. The current-day critical density is $\rho_c = 3H_0^2/(8\pi G)$, so the total current energy density in gravitational waves is

$$\mathcal{E}_{\rm GW} = \int_0^\infty \rho_c c^2 \Omega_{\rm GW} df/f = \int_0^\infty \frac{\pi}{4}\frac{c^2}{G} f^2 h_c^2(f) df/f\ , \tag{2}$$

where h_c is the characteristic amplitude over a logarithmic frequency interval around f, and is related to the one-sided spectral noise density $S_{h,1}(f)$ by $h_c^2(f) = fS_{h,1}(f)$.

We want to relate this to the energy radiated throughout the history of the universe, including the effects of redshifts. Suppose that the number of events between z and $z + dz$ per unit comoving volume is $N(z)dz$. Note that "comoving volume" is the volume that the region in question occupies now; at the time of emission, the volume was smaller. The present day energy density can then be expressed with integrals over the frequency and redshift:

$$\begin{aligned} \mathcal{E}_{\rm GW} &= \int_0^\infty \int_0^\infty N(z)(1+z)^{-1} f_r (dE_{\rm GW}/df_r)(df_r/f_r)dz \\ &= \int_0^\infty \int_0^\infty N(z)(1+z)^{-1} f_r (dE_{\rm GW}/df_r)dz(df/f) \; . \end{aligned} \tag{3}$$

We now have two expressions for the same quantity, and can equate them at each frequency:

$$\rho_c c^2 \Omega_{\rm GW}(f) = \frac{\pi}{4}\frac{c^2}{G} f^2 h_c^2(f) = \int_0^\infty N(z)(1+z)^{-1} f_r (dE_{\rm GW}/df_r) dz \; . \tag{4}$$

This relates the energy density to the current number density of event remnants, and the energy they released over their lifetimes.

This is an extremely general result. It is independent of the cosmology. It is also unaffected by beaming as long as the beams are randomly oriented. If there are multiple types of sources, just add their contributions. This theorem allows us to compute the background due to any given class of sources, at any given frequency. It turns out that the current-day energy density at a given frequency is not even all that sensitive to $N(z)$, assuming that the sources are not concentrated too much at one redshift. In the problem set you will explore the background due to binaries of different types.

A background due to processes in the early universe (say, before the production of the cosmic microwave background) would be very exciting because it would contain information that is unavailable otherwise. In principle, one could see gravitational waves from very early in the universe, because gravitons have a very small interaction cross section. We need to state clearly that, even by the standards of gravitational wave astronomy, these processes are *highly* speculative. One consequence of this is that although it would be extremely exciting to detect a background of early-universe gravitational radiation, a nondetection would not be surprising. The rest of this lecture is based strongly on notes from Alessandra Buonanno (gr-qc/0303085).

Before examining specific possibilities, let's establish a scaling. What can $\Omega_{\rm GW}(f)$ depend on? We know that $\Omega_{\rm GW}(f)$ itself is dimensionless. From its definition, it has a factor of H_0^{-2} (from $1/\rho_c$) and must also be proportional to $S_{h,1}$. It could be proportional to some power of f as well. You might think that a factor of G could also be involved, but since $\Omega_{\rm GW}(f)$ is dimensionless and all the other factors have units of s or s^{-1}, there cannot be any powers of G. The only combination that works is

$$\Omega_{\rm GW}(f) \propto H_0^{-2} f^3 S_{h,1}(f) \; . \tag{5}$$

You will derive the proportionality constant in the homework. The result of this is that unless in some frequency range $d\ln\Omega_{\rm GW}(f)/d\ln f > 3$, the spectral density decreases with increasing frequency. As a result, for most realistic sources of background, it will be easier to detect the background at lower frequencies.

As our first contestant for a gravitational wave background, let us consider simply a thermal background of gravitons. It is possible that in the very early universe (roughly the Planck time, or about 10^{-43} s after the Big Bang!) gravitons were in thermal equilibrium with the rest of the matter and energy. At some point not long after the Planck time, the gravitons decoupled and streamed freely. What would their temperature be now? As a guide, we can consider the cosmic microwave background, at a temperature of 2.7 K. Would the graviton background have a larger or smaller temperature? It would have to be smaller. The way to see this is to realize that at any given epoch, the energy is shared among all the relativistic species that are coupled tightly. At the time of the neutrino background, for example, in addition to photons there were neutrinos, electrons, and positrons. As a result, the temperature that we see now would be less, a bit under 2 K. Since neutrino scattering cross sections scale as the temperature squared, the probability of scattering is some 20 orders of magnitude less than the normal $\sim$MeV neutrinos detected with enormous pools of water or other substances. No chance.

For gravitons, at the Planck time the energy would be shared with the entire zoo of particles in the Standard Model, leading to a present-day temperature of a bit under 1 K. The frequency of the photons would therefore be $\nu \sim kT/h$, or in the $\sim 10^{10}$ Hz range. No current or planned detectors could see this. The whole line of argument leading to this background is in any case dubious, because Planck-era physics is unknown and gravitons might never have been strongly coupled to other particles.

What about generic possibilities from later in the universe? Suppose that a graviton of frequency f_* is emitted when the scale factor of the universe is a_*. Then we observe the frequency at $f = f_*(a_*/a_0)$, where a_0 is the current scale factor. This gives

$$f \approx 10^{-13} f_*(1\ \mathrm{GeV}/kT_*)\ , \tag{6}$$

modulo a factor of order unity related to the number of relativistic degrees of freedom at time t_*. Here T_* is the temperature. What value of f_* should we take? Based on general causality considerations we know that the *lowest* frequency possible is the Hubble constant H_* at that epoch. The frequency could be higher, though, so we'll take $f_* = H_*/\epsilon$, with $\epsilon < 1$. In the radiation-dominated epoch of the universe (valid for $z > 10^4$, which will be true for almost all processes we consider), $H_* \sim T_*^2$, so $f \sim T_*$, or with the constants put in,

$$f \approx 10^{-7}\ \mathrm{Hz}\,\epsilon^{-1}(kT_*/1\ \mathrm{GeV})\ . \tag{7}$$

If ϵ is not too much less than unity, this tells us the energy scale probed at a given present-day gravitational wave frequency. For example, LISA (at 10^{-4} Hz) would probe the TeV scale and ground-based detectors would probe the EeV scale.

What limits are there to the overall strength of the gravitational wave background? One comes from Big Bang nucleosynthesis (BBN). This is the very successful model that relates the overall density of baryons in the universe to the abundances of light elements. The idea is that in the first few minutes of the universe, after the temperature had dropped below the level when photodisintegration of nuclei was common but before free neutrons decayed, protons and neutrons could merge to form heavier elements. George Gamow, who originally proposed this, had hoped that this process would explain all the elements in the universe, but the lack of stable elements at mass 5 and mass 8 prevents this. Instead, just the light elements are formed. These include hydrogen, deuterium, helium-3 and helium-4, and trace amounts of lithium and beryllium. The relative abundances of each depend only on the overall entropy of the universe (i.e., ratio of photons to baryons) and the number density of baryons. Measurements of primordial abundances of the light elements are in excellent agreement with the baryon fraction $\Omega_b \approx 0.04$ measured independently from the microwave background.

If the current energy density of gravitational waves were too high, this would mess up BBN. The constraint is

$$\int_{f=0}^{f=\infty} d\ln f\, h_0^2 \Omega_{\rm GW}(f) < 5\times 10^{-6}\ , \tag{8}$$

where $h_0 \equiv H_0/100{\rm km\ s^{-1}Mpc^{-1}}$. In principle you could imagine that $\Omega_{\rm GW}(f) \gg 10^{-5}$ in some narrow frequency interval, but this seems unlikely.

Observations of the cosmic microwave background (CMB) also limit the GW background. This is because gravitational waves would produce fluctuations in the CMB, hence the measured fluctuations give an upper limit at low frequencies:

$$\Omega_{\rm GW}(f) < 7\times 10^{-11}(H_0/f)^2\ , \qquad H_0 = 3\times 10^{-18}\ {\rm Hz} < f < 10^{-16}\ {\rm Hz}\ . \tag{9}$$

Yet another constraint comes from pulsar timing. The tremendous stability of millisecond pulsars means that we would know if a wave passed between us and the pulsar, because the signal would vary. Roughly eight years of timing has led to the bound

$$\Omega_{\rm GW}(f) < 5\times 10^{-9}(f/f_{\rm PSR})^2\ , \qquad f > f_{\rm PSR} \equiv 4.4\times 10^{-9}\ {\rm Hz}\ . \tag{10}$$

What are some specific mechanisms by which gravitons can be generated in the early universe, after the Planck time? The two primary mechanisms that have been explored are production during inflation, and production during a phase transition.

Various models of inflation have been discussed, but one that is considered relatively realistic is slow-roll inflation. In this model, the universe had a scalar field that, at the beginning of the inflationary period, was not at its minimum. The field value “rolls” towards the minimum and as it does so it drives rapid expansion of the universe. The rolling process means that the Hubble parameter is not constant during inflation. Therefore, fluctuations that leave the Hubble volume during inflation and re-enter later have a tilt with respect to other fluctuations. The net result of calculations is that if standard inflation is correct then, unfortunately, there is no hope of detecting a gravitational wave background, because the amplitude is orders of magnitude below what current or planned detectors could achieve. Variants of or substitutes for standard inflation have been proposed that might lead to detectable gravitational radiation, including bouncing-universe scenarios and braneworld ideas, but whether these encounter reality at any point is anyone’s guess!

If phase transitions in the early universe (e.g., from a quark-gluon plasma to baryonic matter) are first-order, then by definition some variables are discontinuous at the transition. If the transition occurs in localized regions (“bubbles”) in space, collisions between the bubbles could produce gravitational radiation. In addition, turbulent magnetic fields produced by the fluid motion could generate secondary gravitational radiation, but these are weaker. The most optimistic estimates put the contribution at $h_0^2\Omega_{\rm GW} \sim 10^{-10}$, peaking in the millihertz range. This would be detectable with LISA, but don’t bet on it.

A more recent suggestion has been that gravitational radiation could be produced by cosmic strings. Cosmic strings, if they exist, are one-dimensional topological defects. Assuming a network of cosmic strings exists, it would have strings of all sizes and therefore contribute gravitational radiation at a wide range of frequencies. Recently, some work has been done on the possibility that cusps or kinks in cosmic strings could produce beams of gravitational radiation.

If any of these scenarios comes true and in fact there is a cosmological background of gravitational waves detected with planned instruments, this will obviously be fantastic news. However, what if it isn’t seen? That won’t be a surprise, but there has been discussion about missions to go after weaker backgrounds. It is often thought that the 0.1-1 Hz range is likely to be least “polluted” by foreground vermin (i.e., the rest of the universe!). This may be, but it is worth remembering that there are an enormous number of sources out there in even that frequency range, and that to see orders of magnitude below them will required *extremely* precise modeling of all those sources. Either way, whether we see a background or “merely” detect a large number of other sources, gravitational wave astronomy has wonderful prospects to enlarge our view of the cosmos.