1. Written in Cartesian coordinates, a Minkowski spacetime has the line element

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2 \ . \tag{1}$$

Write explicitly the values of the covariant components of the metric tensor, $\eta_{\alpha\beta}$. Also write explicitly the values of the contravariant components, $\eta^{\alpha\beta}$.

The following problems involve manipulation of tensors in the Schwarzchild spacetime

$$ds^2 = -(1 - 2M/r)dt^2 + dr^2/(1 - 2M/r) + r^2(d\theta^2 + \sin^2\theta d\phi^2) \ . \tag{2}$$

Here M is the gravitational mass (that would be measured by observing the orbit of an object at a very large distance), and r is the "circumferential radius", i.e., the circumference of a circle divided by 2π. For particles moving on geodesics in this spacetime, the specific angular momentum is u_ϕ and the specific energy is $-u_t$, and both are conserved.

2. Write explicitly the covariant components $g_{\alpha\beta}$ and contravariant components $g^{\alpha\beta}$ of the metric tensor.

3. Consider a particle in circular motion, with $u^r = u^\theta = 0$. Using the fact that $u^\alpha u_\alpha = -1$ for a particle with nonzero rest mass, derive the specific energy $-u_t$ as a function of u_ϕ. Note that u_ϕ does not have to be the value for a Keplerian orbit. To test your expression, consider a particle on the surface of a nonrotating star of radius R (such that $u_\phi = 0$). Does your expression make sense in the Newtonian limit $M/R \ll 1$?

4. The relativistic equation of motion for a particle with nonzero rest mass is

$$\frac{d^2x^\alpha}{d\tau^2} + \Gamma^\alpha_{\mu\gamma}\frac{dx^\mu}{d\tau}\frac{dx^\gamma}{d\tau} = 0 \ . \tag{3}$$

where τ is the proper time (this would be replaced by an affine parameter λ for a massless particle) and

$$\Gamma^\alpha{}_{\beta\gamma} = \frac{1}{2}g^{\alpha\mu}\left(g_{\mu\beta,\gamma} + g_{\mu\gamma,\beta} - g_{\beta\gamma,\mu}\right) \ . \tag{4}$$

In the Schwarzschild spacetime, derive the radial equation of motion $d^2r/d\tau^2 + ??? = 0$. You may choose your coordinate system so that $u^\theta = 0$.

5. Using your equation of motion, derive the specific angular momentum of a circular orbit (which has $d^2r/d\tau^2 = 0$).

6. Find the radius r at which the angular momentum is a minimum, and the value of the minimum angular momentum. By considering a circular orbit that loses a small amount of angular momentum, show that the radius of minimum u_ϕ is also the minimum radius of a stable circular orbit.

7. Using your expression for the specific angular momentum of a circular orbit, and for the specific energy, to derive the radius of the *marginally bound* orbit, which is where $-u_t = 1$ and hence a slight perturbation outward could send the particle to infinity.

8. The Schwarzschild time coordinate t is the elapsed time as seen at infinity. Therefore, the angular velocity of an orbit as seen from infinity is $d\phi/dt = u^\phi/u^t$. Use this and your previous expressions to derive the angular velocity of a circular orbit at radius r, as seen at infinity.

9. A colleague of yours, Dr. I. M. N. Sane, plans to explore a black hole more directly. His idea is to free-fall radially to a nonrotating $10\,M_\odot$ black hole, then, just outside the horizon, fire his rockets outward to escape. Ignoring the overwhelming acceleration he would feel when he fired his rockets, estimate the maximum tidal force he would feel during his radial free fall and use that estimate to counsel him on whether his trip is advisable.

10. Confirm the expressions for the Christoffel symbols $\Gamma^\alpha{}_{\beta\gamma}$, the Riemann tensor $R^\alpha{}_{\beta\gamma\delta}$, the Ricci tensor $R_{\alpha\beta}$, and the Einstein tensor $G_{\alpha\beta}$ given in the lecture, to linear order in $h_{\alpha\beta}$.

11. Confirm that switching to the trace-reversed perturbation $\bar{h}_{\alpha\beta} = h_{\alpha\beta} - \frac{1}{2}\eta_{\alpha\beta}h$ and applying the Lorentz gauge condition $h_{\alpha\beta}{}^{;\alpha} = 0$ reduces the Einstein equation to

$$G_{\alpha\beta} = -\frac{1}{2}\Box\bar{h}\ . \tag{5}$$

12. Now, impose the transverse traceless condition. Suppose the wave is traveling in the z direction. How many components of the perturbation tensor can be nonzero? After symmetry and tracelessness are imposed, how many independent components are there?